

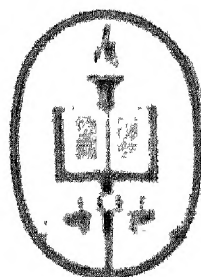
THE PSYCHOLOGY AND TEACHING OF ARITHMETIC

By

HARRY GROVE WHEAT

Professor of Mathematics

State University of New York



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NOTES

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plex world does not improve the number system upon the individual; the individual brings the system, as well as he can, to the complex world and uses the system as a means of bringing order out of complexity. The savage lives in the world a long long time before he invented counting and was able to use counting as a means of giving some system and order to his world of material things, and many children move through the public schools without gaining an understanding of arithmetic because they have been overwhelmed with their sensations of material things and material practical cases.

It is important for the teacher from time to time to take stock of his beliefs relative to the subject for himself, to correct and enlarge his views, and to distinguish between what his subject appears to be and what it actually is. This book has been written in order to describe the present state and characteristics of arithmetic, and to distinguish between its appearance and its actuality, with the intention that once a teacher understands the essential features of the subject he will have some intelligence about holding them up to the attention of his pupils. When the primitive notions gathered after more exact number ideas, how various processes adopted more and more systematic devices for describing and expressing number relationships, to what they finally descended through long centuries of haphazard experimentalism to give their attention in order to bring to perfection the circular system in use today, the intimate relationships between the various parts of the system, and to what the teacher of today must give his attention in order to become acquainted with the system are, accordingly, major topics for discussion. Reference to much of the recent writings about arithmetic as a composite of unrelated skills and about concrete and remedial instruction in arithmetic is conspicuously absent from the present discussion. What the pupil should know rather than what the pupil should avoid is the theme. The discussions of this book are intended to describe the number

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THE PSYCHOLOGY AND TEACHING
OF ARITHMETIC

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The arithmetic that we teach children in our schools today is a number system that has been perfected by the race over a long period of time from very crude beginnings. In order to understand thoroughly our present number systems, it is desirable to have a reasonable acquaintance with the beginnings of numerical thinking among primitive peoples and with some of the main steps that were taken by the race in its passage from these early crude beginnings to our present highly complex science of mathematics.

The actual origin of the concept of number as of course something that we do not know, it is lost in the obscurities that surround our ancestors of long ago. There is abundant evidence, however, that this earliest concept of number was closely related to experience - in other words, that it was essentially concrete, not abstract. We find in the records of early peoples some indications as to the nature of these early number concepts, and we also find in the languages and customs of primitive tribes of the present day like the tribes living in some parts of South America and the Malay Sea Islands, Africa, and Australia, other indications of these early concepts. From such evidence we can piece together the story of number - how it began and how it grew - and at the same time we can clearly see the source of the number system that the human race has finally developed and that modern society now relies upon children in the school to learn.

I. ATTENTION TO THINGS

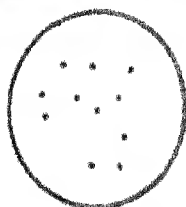
Primitive man lived in a world of things and sensations of these things. He competed for his very existence with other men and with the animals about him. He obviously had to be alert to what was going on about him - at least sufficiently alert to cope with whatever threatened his existence. The evidence shows, however, that from a quantitative (arithmetical) point of view his ideas about nature and his physical surroundings were of a very crude sort. If he was watching

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১৯৭৭ সালের ১০ মার্চ তারিখে বাংলাদেশের প্রথম সংসদ নির্বাচনে বাংলাদেশ জাতীয়তাবাদী দল (বাংলা জাতীয়তাবাদী দল) প্রার্থী হিসেবে অংশগ্রহণ করেছিল। এই নির্বাচনে বাংলাদেশ জাতীয়তাবাদী দল (বাংলা জাতীয়তাবাদী দল) প্রার্থী হিসেবে অংশগ্রহণ করেছিল। এই নির্বাচনে বাংলাদেশ জাতীয়তাবাদী দল (বাংলা জাতীয়তাবাদী দল) প্রার্থী হিসেবে অংশগ্রহণ করেছিল।

Fourth, because we have paid for our transportation well above the minimum known production costs, have had sufficient demand for goods that we have produced, and have managed to keep our costs low, we have been able to produce goods that we have sold at a profit. This is the only way that we have been able to produce goods that we have sold at a profit.

partly appreciate the vagueness of the ideas about groups that primitive man had if we resort to this sensory experiment. Look at three three groups of dots without trying to count them all. You can distinguish a small group, a large group, and a middle-sized group. But just how small, and just how large are these groups? You will have only a rough notion of the size of each group and of their relative sizes unless you stop to count them; mere sensory impression, mere looking, does not reveal the exact sizes; that is, the precise number of dots in each case. If you are thus helpless in your attempt to estimate at all precisely the sizes of these groups,



how helpless must have been primitive man who could not count, who had no names for numbers, and who thus was limited to sensory impression to guide him in thinking about the groups that came within the range of his attention, even though they were groups of as few as seven, five, or even three or two individual objects.

II. GROWING FROM THE NEUMAN IDEA

While the primitive man was engaged in the arduous business of seeking his food or guarding his safety he had no time for reflection; but while he rested in the privacy of his cave, he could pass in review the experiences of the day, could see again in imagination the things he had seen individually and in groups, and could then try to think of one group as compared with another one. He might then want to exchange experiences with his fellows. He may imagine him telling about his experience with this group of animals and that one, and referring to their relative sizes, not as one,

of attention was sufficient for him to keep the things in mind so long as there were not many of these things — no longer as his possessions were few. Each of these things was given a name, the same as each member of a family is given a name. The name made each object a distinct individual and kept it in its proper place in his thinking. The shepherd then early herdman thought of his sheep as individuals. The appearance and the peculiarities of each sheep made it an individual sheep. He gave each one a name and this name served to fix in his mind the individual object. He remembered the names of his sheep, and in that way he would keep track of his flock. A primitive shepherd would not say that he owned "nine" sheep, but he knew that he owned "Whitey" and "Brownie" and "Bogged" and the others that he could name. Thus there is a special significance attached to the biblical allusion to the good shepherd who knew his sheep by name. If he did not know his sheep by names — that is, as individuals — he had lost a major idea of the number in his flock, because he had no other way of knowing the individuals of his flock with any degree of certainty.

Nevertheless, this use of nature to keep track of property units was a clumsy device, a cumbersome means of keeping an account. Naming was not numbering. The names of sheep had no permanence, and they had no fixed order of sequence. Consequently, the primitive man had to turn eventually to other and better means to classify his things or groups.

IV. MATCHING, OR ONE-TO-ONE CORRESPONDENCE

Primitive man came early into possession of another device that led to more accurate, if not more exact, ideas of groups than he otherwise would have gained. This was the device of *matching* the objects of one group with the objects of another group. If both groups were exhausted simultaneously, equality became evident; if one was exhausted before the other, inequality became evident. In situations where

stage of development of his ideas of number wanted to work on a group of things that concerned him, to order others in his memory or to express it with understanding and with no equivocation to his fellows, he matched the objects one by one, not with the objects in any group that happened to be at hand but with those in an appropriate model group. The method of studying the group that concerned him was a method of comparison. Accordingly, his method of thinking about the group, once it had been told off, was also one of comparison. When he thought about his group, or talked about it, he had in mind such comparisons as the following: "I caught as many fish as the witch has sons," "I have as many skins as the clover has leaves." As similar model groups were used again and again in comparisons, they became more and more familiar and understandable -- as we might say, they became standard groups. Any *thema* group that had been compared with a model group became also a familiar and understood one. By means of such comparisons the primitive man developed his ideas of groups toward clarity and exactness.

VI TALLYING

In order to think about and comprehend groups larger than the model groups just mentioned, early peoples continued to employ the method of matching, but with various modifications. Instead of using many model groups as the basis for the matching operations, they employed what we may call ready-made groups, using the objects of these ready-made groups as 'tallies,' or 'counters,' for the objects in the larger groups they wanted to study. The device employed is known as the device of *tallying*.

Our own ancestors must have used the device of tallying a good deal. Our words 'tally' and 'calculate' are borrowed from the Latin *talis*, 'cutting,' and *calculus*, 'pebble,' and take us back to the time when a record of a group was made by cutting a notch in a stick for each object or by laying

Although in the process each object was attended to *one* by one, and recorded by a tally, and although at the end one had a group of tallies that was in a certain way the same as the group being reckoned, still the idea obtained was not complete. The explanation is that tallying was merely a means of substituting one group in consciousness for another. One could think back and forth from one group to the other, recognizing the while that one was the same (in size) as the other. But *how many*, after all, was the tallied group? This remained a question still unanswered. The primitive man could feel them, lift them, look at them, but whatever idea of number he had was an idea based upon mere experience, which remained an idea more closely associated with sensing than with knowing.

Though the tallies of primitive man may have inspired in him some confidence in being able to record and to check his possessions, nevertheless his group of tallies, once they had been accumulated, still remained vague and indefinite, though less vague and less indefinite perhaps than the possessions he was trying to record. The point is that, when he had acquired his collection of tallies, he was unable to think or to name their number. In other words, he could not, like modern man, count his tallies, and so they remained a group that had no special meaning or advantage except what came from ease of handling.

VIII. THE LIMITED VALUE OF MODEL GROUPS

It may very well seem to us that primitive people should have had little difficulty in learning to count their tallies, since they had ready for such use the names of their familiar model groups. The words like 'wing,' 'clover-leaf,' 'octred-toes,' 'hand,' which they used as comparative words in referring to groups of two objects, three objects, four objects, and five objects, respectively, appear to lead easily and naturally to actual counting. We can readily use these names as counting words, thus: *nose, wing, clover-leaf, octred-*

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In Aryan languages there is a difference in kind between the first three or four numerals and the last six or seven. The former are adjectives and are so inflected, the latter are nouns neither in form and uninflected, interjections, as it were, thrust into the sentence in brackets, like the dates in a history book. This difference in kind seems to point to a difference in etymology and also in antiquity. The higher numerals, being nouns, are names of things which are not readily connected with, and subject to, the same relations as the other things mentioned in the same sentence. Secondly, the general abruptness of the transition from low inflected numerals to higher uninflected forms points to some sudden stride in the art of counting. All the facts are readily explained if we conceive that among the Aryans, as among other races, the counting of low numerals was learned before the use of the fingers suggested itself, and that as soon as the fingers were seen to be the natural stores, a great advance in arithmetic was immediately made. The higher unit-numerals would then be the names of the gestures made in finger-counting, or, as among the Algonquins, etc., the actual names of the fingers in the order in which they were exhibited in counting.¹

The paragraph from Gow may be summarized by saying that the use of model groups did not lead to the development of number ideas as employed in counting, but that, after counting began, the names of such model groups came to be useful as the names of number ideas that had been developed by processes somewhat different from those employed in matching objects. To put it in another way, it may be said that sensory experiences did not lead to number ideas, but were useful in illustrating the number ideas after they had been developed through other means.

IX. FINGER-TALLYING

Records of primitive people show that the process of counting was usually related to the use of the fingers as

¹ Gow, *op. cit.*, pp. 9-10.

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Training for the 21st Century

It should go without saying that cooperation with a group is necessary in order that one may have knowledge of it. On the other hand, no one discussion here which would prove experience is insufficient to give one any more than a rather

undefined knowledge of the group. Even when one views and handles each member of the group, tallies each one by one, and finally compares the group with the group of tallies he has accumulated, he still is left with uncertainty. More than sensory experience is needed to clarify and make definite the idea of the group.

The foregoing observations need to be kept in mind, because the notion is prevalent that, wrapped up in the group, is the quality or characteristic that, when brought within the range of one's experience, gives the idea of number. Thus it has been stated that the child acquires the number idea 'five,' for example, by seeing five apples, five boys, five chairs, five pencils, and five of enough other things. Within a group of five objects, we are told is the 'quality of fiveness.' Let the quality once be observed, or let it be experienced frequently enough so that this 'quality of fiveness' is abstracted a little here and a little there from enough groups of five, and the various modicums of 'fiveness' become generalized by the child mind into his idea of 'five.'³ It will be well, whenever one encounters such statements, for one to remind himself that primitive man did not begin to develop and to employ the idea 'five' with other ideas in counting until he had taken at least a step beyond his direct perceptual experiences. Man lived in the world, surrounded by objects that appeared singly and in groups of two, three, four, and so on, for a long time before he began to make a beginning in what we now consider the very primitive process of counting.

Let the child have the number names, and associate them with the appropriate groups, 'two' with a group of two objects, 'three' with a group of three objects, and so on — we are advised — and he will eventually derive the appropriate ideas. Here, again, we must remember the limitations of mere experience, even when it is accompanied by appro-

³ E. L. Thorndike. *The Psychology of Arithmetic* (The Macmillan Company, New York, 1922), pp. 170-172.

prate names. We should recall that primitive man had the model groups and their appropriate names a long time before he developed exact number ideas. However, ~~primitive man~~ His model groups, as we have seen, gave him some of the abstract ideas of number and the ~~means~~ ~~for~~ ~~their~~ ~~use~~ at least, not before he had developed ideas by which he could use them in use in the beginning.

Our discussion to this point ought not to be taken as indicating that primitive man's ~~primitive~~ ~~experiences~~ ~~did~~ ~~not~~ ~~prepare~~ ~~him~~ ~~for~~ ~~the~~ ~~understanding~~ ~~of~~ ~~the~~ ~~ideas~~ ~~that~~ ~~led~~ ~~to~~ ~~exact~~ ~~ideas~~ ~~of~~ ~~number~~. As we shall see he continued to use in his later and more limited study of groups the essential features of the methods we have seen. ~~What~~ ~~we~~ ~~should~~ ~~gain~~, ~~however~~, ~~from~~ ~~our~~ ~~study~~ ~~of~~ ~~the~~ ~~groups~~ ~~of~~ ~~primitive~~ ~~man~~ ~~after~~ ~~some~~ ~~and~~ ~~some~~ ~~groups~~ ~~like~~ ~~the~~ ~~groups~~ ~~that~~ ~~came~~ ~~within~~ ~~his~~ ~~observance~~ ~~is~~ ~~the~~ ~~realization~~ ~~that~~, ~~as~~ ~~long~~ ~~as~~ ~~he~~ ~~depended~~ ~~entirely~~ ~~upon~~ ~~what~~ ~~he~~ ~~could~~ ~~see~~ ~~and~~ ~~hear~~, ~~he~~ ~~never~~ ~~attained~~ ~~an~~ ~~exhausting~~ ~~degree~~ ~~of~~ ~~the~~ ~~feelings~~ ~~of~~ ~~familiarity~~. We should be ~~aware~~ ~~of~~ ~~the~~ ~~limitations~~ ~~of~~ ~~mere~~ ~~experience~~. We should recall that primitive man had to go beyond, or back of, experience before he succeeded in even starting the development of the ideas of number that we know and use with such ~~confidence~~ ~~today~~. We should gather that primitive man was on sound off a device for dealing with his experience.

CHAPTER II

THE BEGINNING OF COUNTING

ARGUMENT

1. The distinctions between counting and other primitive methods of studying groups are brought out in studies of early peoples, such as Conant's. Chief among these distinctions are the following:

- (a) The first counting words were adjectives; that is, 'distinguishing' words.
- (b) The first counting words were related to each other in orderly sequence.

2. A counting word serves the double purpose of fixing attention upon an object in a group both as a single individual item and also as belonging with all the objects that precede it in the counting; that is, it supplies both the *ordinal* and the *cardinal* sense.

3. Until such double-purpose words were invented, or their use discovered, the names of model groups could not become counting words. Such names supplied only *cardinal* number, not the system inherent in *ordinal* number.

4. A word that might become a number name is useless for the purpose without the accompanying *idea* that such use is possible.

5. Counting was extended and new number names were adopted as new number ideas were developed.

6. New number ideas were developed by various methods of combining number ideas already acquired. Old number names were often compounded to form new number names.

7. Such combination and compounding generally followed a *system* that had often been hit upon accidentally.

8. The accident of the fingers served to fix the number base that was used most frequently for combination and compounding.

9. Though separate number names facilitated the de-

violation of new number ideas through combination, and compounding, they had a limited value that was determined by the limited span of attention.

10. The accidental retention of ten, as a number, has done not remove the use of separate number names very far, if at all, from the advantages of direct perceptual experience.

and the other side of the coin

The preceding chapter has called attention to certain methods of studying groups that were employed by primitive peoples and to the nature of the number ideas resulting from the use of such methods. It has indicated that, though these methods of study give the appearance of enumeration, they nevertheless resulted in ideas that were heterogeneous. The number ideas of primitive man were therefore always less dependable; they could not be used with certainty as a basis; they were felt rather than known, they had something to be desired.

1. EARLIER METHODS OF ENUMERATION

Primitive man, as we have seen, numbered the stones of his property; he tallied the sheep by a 'one-for-one correspondence' with his fingers; he enumerated things, 'one by one' with various kinds of talismans, and he thought of these numbers in comparison with the number of objects in various familiar model groups. He then gave attention to each individual item in the given group, and he gave attention to the group as a whole, composed as it was of all the individual items. Insofar as the giving of attention to a group is concerned, that is about all any one can do. There is no other business or quality pertaining to the group in which our own eyes are taken. Why, then, did not primitive man gradually gain such exact number ideas as those with which we are familiar, especially since he might readily have formed the names of his model groups into number names? The only answer that can be given is that some essential was lacking in the methods of study.

II. HOW COUNTING BEGAN

We may, perhaps, gain some notion of the essential that was lacking if we turn to a description of the activities of counting as they first began to develop:

By the slow, and often painful, process incident to the extension and development of any mental conception in a mind wholly unused to abstractions, the savage gropes his way onward in his counting from 1, or more probably from 2, to the various higher numbers required to form his scale. The perception of unity offers no difficulty to his mind, though he is conscious at first of the object itself rather than of any idea of number associated with it. The concept of duality, also, is grasped with perfect readiness. This concept is, in its simplest form, presented to the mind as soon as the individual distinguishes himself from another person, though the idea is still essentially concrete. Perhaps the first glimmering of any real number thought in connection with 2 comes when the savage contrasts one single object with another — or, in other words, when he first recognizes the pair. At first the individuals composing the pair are simply 'this one,' and 'that one,' or 'this and that'; and his number system now halts for a time at the stage when he can, rudely enough it may be, count 1, 2, many. There are certain cases where the forms of 1 and 2 are so similar that one may readily imagine that these numbers really were 'this' and 'that' in the savage's original conception of them; and the same likeness also occurs in the words for 3 and 4, which may readily enough have been a second 'this' and a second 'that.' In the Lushu tongue the words for 1 and 2 are *tizi* and *tazi*, respectively. In Koriak we find *ngroka*, 3, and *ngraka*, 4; in Kolyma, *niyakh*, 3, and *niyakh*, 4; and in Kamtschatkan, *tsuk*, 3, and *tsuak*, 4. Sometimes, as in the case of the Australian races, the entire extent of the count is carried through by means of pairs. But the natural theory one would form is that 2 is the halting place for a very long time; that up to this point the fingers may or may not have been used — probably not; and that when the next start is made, and 3, 4, 5, and so on are counted, the fingers first come into requisition. If the grammatical struc-

ture of the earlier languages of the world a lesson is suggested to the student in search of the germination of the Sanskrit system in them -- something which tends to strengthen an impression undergone extended development. The Sanskrit system exists unequivocally in the time when 1 and 2 were the primitive at mankind's disposal, to the time when the Sanskrit system, 1, 2, many, each denoted a distinct aggregate. With increasing knowledge the community for the differentiation would pass away, and last from masculine, singular and plural, would remain. Incidentally it is to be noticed that the Indo-European words for 1 *ekam*, *dvam*, *trayam*, *catvaram*, etc., have the same root as the Latin *primo*, *secundo*, and *tercio* as a hint of the time when our Aryans were first introduced to this manner I have just described.

The first real difficulty which the average mathematician in counting, the difficulty which occurs when he attempts to pass beyond 2, and to count 3, 4 and 5, is of course not slight and these numbers are everywhere much hard words, and are not by almost all tribes, as numbers have deeply sunk in the language we find them. But the evidence that this is actually the case cited must not be forgotten. The languages of even the most primitive state, properly count all up to the hundreds like the Vedas, and many of the American tribes have an hundred higher than 2, others of the American tribes even reach 1000. South American tribes with 2 on 2 and others which reach 5 their limit are still more numerous. Hence it is not so much that even the uneducated mathematician is not only a beginner but a perfect one. Beyond 5 perhaps some of the primitive words the greatest difficulty.

III CHARACTERISTICS OF THE VEDIC

From Cassini's description of the languages of the Vedas we can discern certain characteristic features of the Sanskrit language.

1. The First Counting Words Were Indistinguishable Words

The first that strikes the attention is that the first counting words were used as imperatives in the Vedas, as pointed out.

From I. I. Cassini: *The Language of the Vedas*, pp. 1-11. The publication of The Marathi Language, published by the Government of India, 1900.

the objects to which they referred. By the use of such words, the primitive man gave his attention to the objects in a group in a new and different way. Note how his first words for one and two were such as to give the sense of 'this' and 'that.' These were *distinguishing* words. They made the objects, to which they thus referred, stand out each as distinct from the other. They separated the objects so named quite as much as our words, 'this one' and 'another one,' or 'first' and 'next,' or 'first' and 'second.' Even though the words were able to carry counting only as far as two — or, by the use of somewhat similar words for 'another this' and 'another that,' as far as four — they moved the ability to give attention to the objects of a group a step in advance of what it could ever have been by the use of tallies. To be sure, one could say "finger," "finger," when he tallied on his fingers, and each finger could be used to designate an object tallied, but the words, 'finger,' 'finger,' carried no sequence. The words 'this' and 'that' were *sequential*; 'that' must follow 'this' in ordered sequence; 'another this' must follow 'that,' and so on. 'This' always came first; 'that' came next. There was no mistaking the order of succession, once it had been established; whereas, when fingers or pebbles were used as tallies, it made no difference in what order the tallies were used. The tallies were for all practical purposes all alike. There was no order, and no established sequence among them.

2. Counting Words Established a Sequence

The second feature of early counting that strikes the attention is closely the ally of the first mentioned; yet it is distinct from the first. The second feature is that, since the counting words established a sequence, each succeeding word took into account and depended upon all that preceded. Each counting word thus had a place in the series that was both determined and fixed by the preceding words and that helped to determine and to fix the place of the fol-

lowing words. Such a feature of the counting words served not merely as a means by which each object represented was distinguished as an individual item of attention, but also as a means establishing in correspondence the extent of the group that was being counted. Because of the inherent association of the counting words, each word served a double purpose. It both called attention to the individual items of the group to which it particularly applied and also called attention to this item in connection with all the things that had preceded. The first feature gave attention to individual items, and the second gave number its numerical sense. And because both features were determined by the ordered sequence in which the counting words came, each word was used at one and the same time in the double sense. Individual and numerical numbers were thus for the first time created or thought.

IV THE COUNTING SYSTEM

The source of counting is its objects. It proceeds, just as tallying and matching with marked groups provide information both to the objects of a group one by one and to the group as a whole. It does more. It provides a system. Through the activities of tallying and matching concretely presented to the primitive mind for the emergence of abstract concepts formed in connection with counting, the system that they gave into counting would constitute a presentation of the true meaning of counting. In order to provide the presentation of counting, the savage mind had to have at its disposal something more than a hasty array of marked groups. In order extensive the array may have been. It had to contain an order and a system in its procedure of attaching its objects and to groups. This order and system which grew as counting grew, constituted the beginnings of classification. At this point, there was no mathematical thinking and the system mathematics started to develop.

We can now begin to see what the earlier methods of studying groups lacked. We can observe the common thing

the primitive mind did not immediately proceed to the process of counting as soon as it had learned to resort to the use of words for model groups, such as 'wing,' 'clover-leaf,' 'ostrich-toes,' 'hand.'

Counting is a complex procedure. One must have the number names in serial order ready for use. One must discriminate the objects being counted; that is, one must attend to each object singly, one at a time. One must apply the number names to the discriminated objects. And one must think the whole number of objects counted together as a group. Thus, in counting: one, two, three, four, we apply the name 'three' to the third object both as a means of setting it off from all the others and as a means of grouping it with those that have preceded it in the counting. At one and the same time 'three' means *third* object and *three* objects. When one counts his fingers, stopping with the thumb, he calls the thumb *five*. In this sense 'five' means *fifth*; it is an ordinal numeral. At the same time 'five' as applied to the thumb means that the thumb, combined with the rest of the fingers, is actually *five*; it is then a cardinal numeral. Thus, at one and the same time the counting word applies both to the single object and to the group.

In developing a concept of number in a collection by a one-to-one comparison with a model group, such as the fingers on his hand, for example, the savage used his word 'hand' in the cardinal sense only. He matched the objects being checked off with his fingers. Each finger checked an object. In the process he needed no separate names in a series to check the objects. He used a name — 'hand,' let us say — when the matching was complete. The name was used to designate the group as a whole. In the same way, he could check off four objects by matching with the toes of an ostrich, the picture of which he carried in his mind. To check one group, he used his hand; to check another, he used the toes of an ostrich. Since he used such model collections at separate times to tally first this group, then that

V. IDEAS BEFORE WORDS

After primitive man began to count, he did in many instances use for his number names words that originally referred to concrete objects. Thus, many tribes used the word 'hand' for five, 'man' (all the fingers) for ten, and 'whole man' (fingers and toes) for twenty. The significance of such words, when they were used as counting words, was determined by the process of counting that had previously been created. It can hardly be said that such words, including 'wing,' 'clover-leaf,' etc., led to counting or assisted the primitive man in learning to count. There is a wide difference between our own ability to convert such words into counting words and the ability of the savage *before he had created the idea of counting*. We ourselves have the key that makes the conversion easy. The savage did not come into possession of the key until he had created the idea of using words in a series to designate objects one by one. When he had conceived the idea, he could take over and use as counting words any such names as suited his purpose.

We may summarize our whole discussion to this point in the statement that it takes an idea to make a word significant. Without the idea, the word is neither useful nor suggestive; with the idea, the word becomes serviceable and meaningful.

VI. NUMBER NAMES AS COUNTERS

The development and use of number names as counters further illustrate the development of number ideas as well as certain of their unique characteristics. Let us return to a paragraph from Conant's discussion of the number concept for an illustration of the early use of number names.

In certain parts of the world, notably among the native races of South America, Australia, and many of the islands of Polynesia and Melanesia, a surprising paucity of numeral words has been observed. The Encabellada of the Rio Napa

point he cannot go because of the difficulty of holding the combinations in mind without verbal aids.

In the course of time new names are invented for the new ideas — *three, four, and five*, let us say. The new names for the familiar ideas extend the possibilities of number thinking. Now the primitive man can count, *one, two, three, four, five*, and by means of combinations made possible through the new names, *five and one, five and two, five and three, five and four, five and five, many*.

And so through the centuries, as primitive man developed new number ideas through a combination of his old ideas, he developed new names and new combinations of names to stand for his ideas. He developed his ideas in the course of experience, making use of what meager number names he already possessed. The names he possessed helped him to clarify his ideas and to free his thinking for the development of new ideas. Through comparison and grouping of objects, new ideas were developed; through the use of language to express the ideas, the methods of grouping were facilitated. We can distinguish a double development, a number language and number ideas developing concurrently.

We have seen how the use of pebbles as counters assisted early people in thinking by making possible easier and readier grouping. We can see how the use of number names as counters facilitated grouping by helping to make it a matter of thought without constant reference to the objects to be grouped and without the use of objects as counters, which are not so easy to handle and to move about as names.

VII. COUNTING AND THE NUMBER BASE

Counting was created by primitive man through his efforts to get a clearer and fuller idea of the group. It lent itself to the study of groups and was used for that purpose. By means of counting, attention was given systematically to each object of the group both by itself and also in connec-

tion with all the objects previously mentioned as being at the conclusion of the counting there were included on that day? The idea of the group and the idea of the words included that count, posed it.

As counting came into use as a means of getting systematic attention, first to the groups, then to them, the ideas of the groups counted came to be more and more homogeneous, more and more definite, and hence more and more undifferentiated and usable. Let groups be counted that is included in counting — one and over, and the ideas will come to be used with more and more certainty and assurance through the ability to count is extended only so long as the idea of the ideas to four or five seems to be something they could turn over to all sorts of applications, and hence in their minds the possibility of all sorts of combinations. The experience of feelings of familiarity and assurance may lead to the counted, though they apply to numbers which are farther up the scale than four or five. The experience would be familiar with those ideas which grew out of the earlier efforts at counting that he would be included in the same manner to combinations of the ideas than in the effort to extend the separate number names farther up the scale.

It may thus be seen that counting served a double purpose. First of all, it made possible a systematic study of a few small groups which resulted in definite ideas of number ideas. Secondly, with the development of these number ideas, the process of counting, limited as it was to the groups to which the ideas belonged, in their attention upon these ideas that they were employed over and over at the later study of larger groups.

The extent to which counting was restricted in the beginning determined, at least for a period, the number ideas which were used as a base for the study of larger groups. The development quoted paragraph from Constant's studies has shown. Children dealt with a group of four by thinking of it as three and one and how they dealt with a group of five by thinking of an

as *three and one and one*, or as *three and two*. The comprehensive studies of Conant indicate such number bases as *two, four, five, ten, and twenty*.³

The most prominent of the number bases are *five* and *ten*, which are to be explained by the five fingers on a hand and the total of ten fingers. The fingers served to set the limits of straightforward counting without resort to combination both in the counting systems of primitive peoples and in the counting system in common use today. One learned to count by the use of separate names to the extent of a certain group — two, three, four, five, ten, or twenty, as the case may be — and then, holding the group in mind through repetitions of the name, proceeded with his counting as he began.

The following account of the number system of the Zuni Indians is illustrative. Their number words began:

- | | |
|-------------------|------------------------------------|
| 1. <i>topinte</i> | taken to start with. |
| 2. <i>kwilli</i> | put down together with. |
| 3. <i>hai</i> | the equally divided finger. |
| 4. <i>awile</i> | all the fingers all but done with. |
| 5. <i>opte</i> | the notched off. |

Compounding now begins:

- | | |
|-----------------------|---|
| 6. <i>topalikyā</i> | another brought down to add to the done with. |
| 7. <i>kwillilikyā</i> | two brought to and held up with the rest. |
| 8. <i>hailikyē</i> | three brought to and held up with the rest. |
| 9. <i>tenalikyā</i> | all but all held up with the rest. |
| 10. <i>astemthila</i> | all the fingers. ⁴ |

VIII. THE DEVELOPMENT OF NUMBER NAMES

Foregoing paragraphs have indicated the value of separate and distinct number names in clarifying the thinking

³ Conant, *op. cit.*, Chaps. V-VII.

⁴ F. H. Cushing. "Manual Concepts." *American Anthropologist*, V: 1892, p. 289.

C. H. Judd. *Psychology of Social Institutions* (The Macmillan Company, New York, 1926), p. 84.

of primitive people about related matters where the original name helps to set the idea off as a distinct entity and provides for its verifiable use, such organizations need no combination with other ideas. The act of reorganizing the name for a larger group out of the masses of sections shows that change is no doubt led in many instances in the development and use of a new scientific name for the larger group.

The illustration of compounding taken from the phonetic system of the Zuni Indians suggests how such names may have resulted from the original activities of combination and compounding. Whenever a name for an object, or group of objects — admissible, 'all the fingers' — or the result of an act of grouping — *haukitye*, 'those brought to and held up with the red' — was borrowed for another usage and adopted as a number name, there arose the necessity of distinguishing between the sound of the word when it referred to the object or activity and the sound when it was used as a number name. In the process of repetition this distinction would often become so marked that an observable connection would remain between the two sounds of what was originally the same word. The number names in order to be a real and serviceable number name had finally to lose all connection with the concrete object or activity from which it first came. As the number developed and developed, the number name came to refer to it, and to it only. With the number idea the number name was developed. In the end, the idea, there was no number name.

One can distinguish in the development of marketing the tendency to develop a separate name for the class of each new group as it came clearly and gradually into consciousness as a distinct idea. All sorts of eponyms and group names were resorted to in distinguishing the larger groups and all sorts of sub-groups. A number of names were used. The marketing of the 20th century was one sort of eponyms. Culture and environment

One other method of combination, that of subtraction, remains to be considered. Every student of Latin will recall at once the *duodeviginti*, 2 from 20, and *undeviginti*, 1 from 20, which in that language are the regular forms of expression for 18 and 19. At first they seem decidedly odd; but familiarity soon accustoms one to them, and they cease entirely to attract any special attention. This principle of subtraction, which, in the formation of numeral words, is quite foreign to the genius of English, is still of such common occurrence in other languages that the Latin examples just given cease to be solitary instances.

The origin of numerals of this class is to be found in the idea of reference, not necessarily to the last, but to the nearest, halting-point in the scale. Many tribes seem to regard 9 as 'almost 10,' and to give it a name which conveys this thought. In the Mississaga, one of the numerous Algonquin languages, we have, for example, the word *cangamei*, "incomplete 10," for 9. . . . The same formation occurs in Malay, resulting in the numerals *delapan*, 10-2, and *sambilan* 10-1. In Green Island, one of the New Ireland group, these become simply *andra-lua*, "less 2," and *andra-si*, "less 1." In the Admiralty Islands this formation is carried back one step further, and not only gives us *shua-luca*, "less 2," and *shu-ri*, "less 1," but also makes 7 appear as *sua-lolu*, "less 3." Surprising as this numeral is, it is more than matched by the Ainu scale, which carries subtraction back still another step, and calls 6, 10-4. The four numerals from 6 to 9 in this scale are respectively, *iwa*, 10-4, *arawa*, 10-3, *tupe-san*, 10-2, and *sinepe-san*, 10-1. Numerous examples of this kind of formation will be found in later chapters of this work; but they will usually be found to occur in one or both of the numerals, 8 and 9. Occasionally they appear among the higher numbers; as in the Maya languages, where, for example, 99 years is "one single year lacking from five score years," and in the Arikara dialects, where 98 and 99 are "5 men minus" and "5 men 1 not." The Welsh, Danish, and other languages less easily accessible than these to the general student, also furnish interesting examples of a similar character.

More rarely yet are instances met with of languages which

make use of substantives adjoined as *forms* or *modifiers*, as the composition of numerals. Whether the point here pressed is that an instance has been noticed in the case of the Eskimo language of British Columbia. In their numerical series 1, "one foot," is followed by 14, "one man has a 2," 15, "one man has 3," 16, "one man has 2," 17, "one man has 2," 18, "one man," Twenty-three is "one man and two hands," 24, "one man and two hands has 4," 25, "one man has 2" and so on. This method of formation pervades throughout the entire numeral scale.*

IX. THE LIMITED VALUE OF SEPARATE NAMES

Though the lack of a system of grouping and of compounding names served to facilitate the development of separate and distinct number names that would very readily lose their earlier relationships, it failed to promote corresponding progress in the development of clear and distinct relational ideas. The failure is to be explained by the fact that the span of attention is limited. The mental enumeration difficulty in keeping a multitude of numbers so carefully kept and so distinguishing large groups and in apprehending their exact relations in thought. Thus it requires almost blind faith to separate name from attention upon a large group, as to separate item of experience, it may come to be lost instead of bringing out, the relations between that group and relational groups.

A word is useful and valuable only when the meaning that it is intended to carry is clear and unambiguous. When the meaning is involved, uncertain, or fuzzy the word not only is difficult to use but also misleads and engenders misunderstanding. Thus, to be useful, a number name that is separate and distinct must stand for an idea of a group that can be readily apprehended as for an idea that can be immediately brought out a great deal of preliminary thinking. The more the quality for the ideas to be brought out is pronounced the greater the quality.

* Combs, op. cit., pp. 25-26.

Primitive man, as we have seen, learned to distinguish groups up to three and four by means of direct perceptual discrimination. In a group of four, for example, it is possible to distinguish at once each individual member of the group and the group as a whole. Civilized man, despite his elaborate number training, cannot go much beyond a group of four. With all of our experience we cannot immediately apprehend a group larger than six or seven. However, we are able by a single act of grouping to recognize a group of nine or ten. We can do it by noticing at once five here and four there beside the five, for example. Groups of twelve, fifteen, and sixteen require a double or a triple act of grouping. We learn, to be sure, to think of sixteen as eight and eight, or as ten and six, but this does not make 'sixteen' highly serviceable as a separate and distinct idea, for the reason that sixteen is learned through the combination of ideas that are themselves the products of combination. A number idea, to be serviceable as a distinct idea, must relate closely enough to direct experience to stand in thought as a distinct idea. The ideas, 'one' to 'four' or 'five,' are distinct because they may be related to readily comprehensible groups. The ideas, 'five' to 'nine' or 'ten,' are also learned and used as distinct ideas because they may be learned as applying to groups that are readily comprehensible by an act of grouping that is only one step removed from direct experience. Ideas beyond those of nine and ten may not be considered useful as distinct ideas because their conception and development involve mental processes of grouping too many steps removed from direct experience. As we shall see, they are learned as ideas related to more easily developed ideas, and they are so used in our number system.

CHAPTER III

COUNTING DEVICES

ABSTRACT

1. The use of objects as counters was both an aid to counting and a means of recording the result of the process. It helped to extend the process beyond the limit of the memory span.
2. The use of objects facilitated the division of a large group into smaller groups each corresponding in size to the number base.
3. As number names were associated with counting objects of whatever sort, so they became used as counting groups.
4. Since objects as counters and recorded numbers as counters were used together, the use of a number name to designate a group suggested the use of an object to designate a group. Objects thus became markers of groups as well as counters of the individual items of any given group.
5. When the same kind of counters was used for the double purpose, it was necessary to distinguish between the counters that stood for individual items and those that stood for groups.
6. The position in which the counters were placed was hit upon as the means of keeping these distinguishing functions as a sign of value, that is, as a sign of the size of a group, in a counting device of several stages.
7. The abacus was an adaptation to keep these and distinguishable the positions of counters that were used for different purposes.
8. The abacus made possible an easy reckoning with the size of groups as well as with their number. Size was signified by position and number by the counters.
9. The abacus set off positions where values were assigned regularly in the decimal scale.

10. The abacus took care of the sizes of groups so well that the computer could devote most of his attention to the manipulation of his counters.

11. Interest in the manipulation of counters led to modifications of the abacus that departed from the decimal scale. Later discussions will indicate the results of such neglect of the original significance of position.

12. The suggestion is made that the interest in number manipulation today frequently serves to distract from the significance of position as a means of representing the idea of size.

I. OBJECTS AS COUNTERS

Fingers were the first counting device. Used as tallies before counting began, they were ready for use in finger-counting as counting was created. Finger-counting served a double purpose. In the first place, it helped to inspire a feeling of confidence in the one whose counting activities were just beginning. The use of the fingers in the set order that had been hit upon accidentally or copied from another helped to keep the counting words in order. Since counting words were often derived from the names of fingers or of certain uses of the fingers, the association persisted. Primitive man sought the fullest possible reactions of meaning from his counting activities. He relied upon all the help the use of his fingers seemed to give him as he groped his way toward a better and better use of his number names.

Moreover, when the counting of a group had been completed, the fingers that had been told off in the process of counting stood as a record of the group. Matching, as in tallying, continued, and so the group of fingers was recognized as the equal of the group of counted objects. One could hold up so many fingers to indicate the size of the group. These he or another could count to confirm or to get in mind the idea of the group indicated.

As fingers were useful both as an *aid during the process of*

counting and as a record when the particular kind of object is completed, so were other objects, such as pebbles, sticks, marks in the sand, etc. A pile of pebbles, accumulated during the process of counting a group, constituted a record for the group and to aid the memory in retaining the idea of the group. When the number of the group had become forgotten, or if it had become confused, one would always return to the pile of pebbles and count them.

II. NUMBER OF HIGHER ORDER

The use of objects in counting at the beginning seemed quickly to suggest that one at a moment of passing counting to a higher level. Through them was the foundation base that at first was merely the supporting power of the counting activity because the point of departure had been achieved in counting. To illustrate let us imagine a primitive chieftain who has twenty-five warriors in his band and who has number names only to five. Using the number he has learned to five, proceed with five and one and so on. The man recognizes his confusion when he grasps that he has five more and one of five will reflect that his idea of groups such as five is not quite so altogether perfect and that he probably desires to group them keeping his ideas and names straight even to that point. Let us watch him, however, as he uses objects to assist his counting. As the warriors pass by in the procession he lays aside an object for each until he has counted to five and accumulated a group of five, he perceives no further counting to five, and accumulates another group of five. Finally the counting is completed, and he has counted the five groups. By means of the objects, the counting was extended to groups far beyond the point that would have been possible if nothing but the name, one, two, three, four, five, had been used.

When the chieftain recalls the groups of five he takes a step in advance; he raises the counting activity to a new level; he begins to deal with numbers of higher order. As he counts the groups - one, two, etc. - he is extending

neither warriors nor objects that stand for warriors. *One* no longer means merely *one*, whether warrior or object. *One* now means *one five*. In the final counting, groups are treated just the same as objects — dealt with, spoken of, counted, each as a unit, though it is perfectly clear that each group is unlike the units of which it is composed.

When the savage had begun his counting, he developed and used it as a process having general applicability. He could count men, trees, spears, animals — anything. Thus, when he learned to set aside a group as an understood and manageable unit, he could just as easily count groups. From earliest times large and unwieldy groups have been divided into small and comprehensible groups. Sheep were divided into herds, armies into bands or companies, skins into bundles, and so on. When the division had been made, each herd, or band, or bundle, was treated as a unit and was counted just as readily as the unit items that composed it. When the division was made through preliminary counting into groups of the same size, the groups took on an added meaning and the counting of the groups, a deeper significance. Thus, five groups of five meant a great deal more than five groups of different sizes. Moreover, when five or ten had been used over and over as the basis of divisions and groupings, a group of this particular size became better understood than a group of any other size. Such a group became with usage a *standard* of counting, grouping, dividing, measuring, comparing.

III. OBJECTS AS COUNTERS OF GROUPS

If we were to represent graphically the grouping of objects as the primitive chieftain used them in counting twenty-five warriors, as illustrated in the foregoing topic, we would picture *five fives* thus:



Or, as objects used as counters later came to be grouped, one would have the grouping pictured thus:

|||| |

that is, the fifth stick, or mark, on a given group, was placed across the rest to indicate the completion of the group and to make it easily distinguishable from any collection which was completed group.

As early man learned to attend to groups in the same way as he had attended to individuals, and to apply to given groups the same number names as he had applied to individuals, he gradually came to the point where he found it just as easy to represent a group with an object as it was to represent an individual with an object. Objects were gradually put into use as counters for groups as well as individuals for individuals. If the same names can be used in counting groups as in counting individual units, why could not the same objects be used as counters for both?

The procedure may be illustrated by the way that is often made of the fingers in counting. One often uses the fingers of one hand, the left, for example, to count the individuals until a group of five is counted, and at that point turns down a finger of the right hand to stand for the group of five. He then proceeds as before, and with each group of five uniting off on the fingers of the left hand, he turns down a finger of the right hand. Suppose, when the counting is completed, four fingers on each hand are turned down. One does not think of the two sets of fingers as having equal value. The group counted is not four and four but four tens and four. Each finger turned down to take every other finger is as sure; but the places where the fingers are turned down make a difference.

Let us picture the ancient chieftain again, using objects to count his warriors, supposing that later than he has thought four to count. He counts by the laying down of objects for each warrior. When the last warrior is counted, he lays down

an object to stand for the group, and so on until the counting is completed. At the end, he has accumulated objects in groups thus:

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He now counts his counters: *four* and *four*; but in the one case, each object stands for a warrior, and in the other, each object stands for a group. Instead of having counted only *four* and *four*, the chieftain has counted *four fives* and *four*.

Conant¹ reports a curious method of counting soldiers that was observed by travellers in Madagascar more than a century ago. The soldiers were made to file through a narrow passage, and one pebble was dropped for each. When a pile of ten pebbles had accumulated, a pebble was set aside to stand for the ten and the counting was continued. When ten pebbles had accumulated in the second pile, a pebble was set aside in a third location to stand for the ten tens or the hundred, and so on until the entire army had been counted.

IV. POSITION AS A SIGN OF VALUE

It was of paramount importance, when objects were used both as counters of individual units and as counters of groups, to distinguish between the objects that were used for the different purposes. Suppose, after laying aside so many pebbles to stand for the groups of ten and so many to stand for the remaining individual soldiers, the enumerator of the army in Madagascar had forgotten which pebbles stood for groups and which stood for soldiers. His confusion would have been without limit. Instead of aiding him to arrive at a definite idea of the number, his counters would have led either to a very erroneous and undependable idea or into hopeless confusion. It was necessary, when

¹ L. L. Conant. *The Number Concept* (The Macmillan Company, New York, 1896), pp. 8-9.

objects were used as exemplars for the different groupings, to keep the objects separate and distinguishable.

What criterion might be used in doing this? The objects used as exemplars in this way "were brought out in pairs of different colors and the words which the learner uses in each case may be, and blue for the first and red for the second. The objects of different colors would provide the means of pointing out an intermediate case for the purpose of showing the learner on. Either criterion will serve the purpose but either criterion requires the construction, after examining the objects the learner uses as exemplars, of some definition in the groupings in which the objects are used for distinguishing. It would have to keep the distinguishing features of the exemplars clear in mind; and then would make definition of each grouping the treating of all the exemplars according to a common standard. Moreover, the colors might have to be used as a means of pointing out for one use brought out for another use. It would have to be used as a criterion for distinguishing and for pointing out the objects as a criterion for distinguishing and for pointing out the objects as a criterion for distinguishing.

The use of the fingers of the two hands as a means of both units and groups suggests the criterion that is to be used to manage and thoroughly understand. The fingers of the two hands are separately brought. They are not brought together or confused. They are brought out separately by the learner and are kept clear without preliminary pointing out or preliminary thinking. They are brought out in one case and of different colors and units and the other and are brought out of the same mind is required of all mental for distinguishing. The two sets of exemplars. The exemplars are as such as are brought out treated alike because function brings them to a common level.

Whether the use of the fingers suggests the idea of making use of position in doing this, or whether the idea of position were used for different purposes or whether the idea of position by accident, there is no question of the fact that the fact is that position had come to the mind and to the learner's

indicating value. When the enumerator of the soldiers in Madagascar had used his pebbles as counters to ten, he put aside in another position a pebble that stood as a counter of a group of ten. It was one pebble just like every other pebble but, because of the position in which it was placed, it stood for fifty. The enumerator kept the positions of the counters in turn helped him to keep his mind clear, and his thinking clear.

V. THE ABACUS

The enumerator of the soldiers in Madagascar, and others like him, had during the process of enumeration not only to use their counters, but also to decide upon and to keep in memory the places where they would put the counters of different values. In the beginning there was no method or scheme for the placing of counters and used. The places were determined upon and used according to the dictates of individual and momentary caprice. Counters for the soldiers could, for example, be placed at one's right hand; counters for groups of tens at one's left hand; and counters for tens of tens, or hundreds, immediately in front. Any placing of the counters would serve the purpose. All that was necessary was to keep the positions chosen for units, tens, and hundreds constantly in mind, and not confuse one chosen position with another.

Chance selection of positions for the placing of counters gave way to an ordered scheme. This was effected through the development and use of a mechanical contrivance known as the abacus.

The abacus was originally (1) a dust covered board upon which figures could be marked with a stylus and erased with the finger when necessary, whence the name 'abacus,' from the Greek *abax*, dust. It later was developed into (2) a table marked with lines upon or between which loose counters were placed; and (3) a table or frame on which the

counters were kept in position, or fastened, by means of grooves, wires or rods. Many forms of the abacus have been developed. They all may be classified, however, according to the three general types.

A form of the abacus used by the Egyptians is shown in the accompanying illustration. The counters were fastened on a wax tablet. As many counters as were desired could be utilized. A counter in the first column represented one unit; a counter in the second column represented one ten; and so on; and a counter in any column represented ten counters in the next column to the right. The number 1737 is represented in the illustration.

M	C	D	U
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.

The simplest form of the bar abacus may be represented as shown below. Each bar gave a place value in the number

—————	Thousands	are placed upon a black line
—————	Hundreds	had a value ten times the
—————	Ten	value of the bar below it. The
—————	Units	number 1737 is represented in
		the illustration.

The Russian abacus is perhaps the simplest form of the frame abacus. This form employs beads as counters of value. The string of beads at the right represents units, the next tens, and so on. The beads are set to show 1737.

Each form of the abacus employed the same principle in representing number; namely, the principle of position. In the figures shown the importance of position is shown. Each counter is like every other counter, but each column or group



of counters is unlike every other column or row. A unit counter in unit's column or row stands for a unit, but a unit counter in ten's column or row stands for a group of ten units. In each illustration seven counters are shown in hundred's position and seven counters in unit's position. What each set of seven shows is determined quite as much by position as by the number of counters used.

1. Counting on the Abacus

Because the abacus kept position clear, it was a useful device in counting. The enumerator of soldiers, or of items of property in Ancient Egypt, Babylonia, Greece, or Rome would lay down a counter in unit's column for each soldier, or item, until he came to nine. Now when he counted the tenth one, he would remove the counters in unit's column and place a counter in ten's column, and so on. When he had counted to the point where he had nine counters in each of ten's and unit's columns, he would place the next counter in hundred's column and remove the counters in the first two columns.

C	X	I
		•
		•
		•
		•
		•
		•
		•

2. Operations on the Abacus

The abacus was used as a means of computation by early Egyptians, Babylonians, Greeks, and Romans. Herodotus testifies to the use of the abacus by the Egyptians, saying that they "write their characters and reckon with pebbles, bringing the hand from right to left, while the Greeks go from left to right."³ Smith mentions evidences of the early use of the abacus by the Ancient Maya civilization in Yucatan.⁴ The abacus was used as a computing device in Medieval Europe, and it continues in use today in China, Russia, and Persia. The Chinese laundryman in America may frequently be observed to carry on his computations with an abacus in the form of beads on a wire frame.

³ Smith, *op. cit.*, p. 160.

⁴ *Ibid.*, p. 45.

Addition was a straightforward process corresponding to methods used in counting. It is illustrated by Smith as follows:



ADDITION ON THE ABACUS

An early computer, wishing to add 22 and 139, might have proceeded as follows. Place 2 pebbles on the first line, as shown in the First Step. Then place 3 more, as shown in the Second Step. Then take away 10 of these pebbles and add one pebble to the ten's line, as shown in the Third Step. Then add 2 pebbles to the ten's line because of the 20 in 22, as shown in the Fourth Step. Then add 3 more because of the 30 in 139, as shown in the Fifth Step. Finally draw a line for hundreds, and in this place one pebble because of the 100 in 139. The answer is 161.

Subtraction was a 'take away' process, the opposite of addition.

Multiplication was a process of repeated addition by doubling, and division was a more difficult operation most frequently performed by continued subtraction and a method of completing the division to a multiple of ten. For example, 132 by 5, additions were performed as follows:

132	264	328
132	264	132

To divide 660 by 132, subtractions would be made like:

660	328	396	264	132
132	132	132	132	132

Counting the times 132 has been subtracted, was made the answer 5. These operations were performed on the abacus.

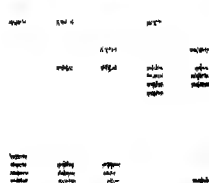
without the use of the Arabic numerals. We have used them here merely to indicate the kinds of processes that were performed.

3. Variations of the Abacus

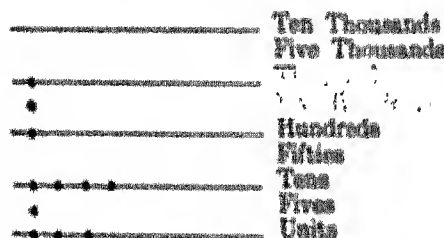
For the sake of facilitating the computations to be performed on the abacus through a reduction of the number of counters to be used, the simpler forms of the device were complicated with counters for fives and multiples of five. Such was the development of the abacus in China, Japan, and Medieval Europe. The principle of position remained



PLAN OF CHINESE
"SUAN-PAN"



PLAN OF JAPANESE
"SOROBAN"



PLAN OF MEDIEVAL COMPUTING TABLE

unchanged in the Chinese and Japanese abaci, but was complicated somewhat in the line abacus of Western Europe by the use of intermediate positions for fives and multiples of five. The developments are illustrated above. In each of these three illustrations the number 1648 is represented.

The illustration of the line abacus shows the form that was used in Western Europe for several hundred years. To

of the group to be represented. Position came to be resorted to in order to distinguish groups from units and from each other. *Size* was represented by position and *number* by the counters.

Thus, in gaining an idea of a large group of objects, the total group was broken down into smaller related groups; rather, the individual units of the large group were built into a number of smaller related groups. To get in mind and to keep in mind the values of the smaller groups in their relations to each other and to the large group being dealt with, the number of the smaller groups had to be accounted for and the size of the smaller groups had to be definitely and accurately represented. Ideas of *number* and ideas of *size* thus developed as complementary ideas. They both had to be dealt with at once. The number of groups was important; the size of the groups was equally important.

The abacus was a useful device in representing sizes by position; it represented related sizes by a systematic scheme of positions for the counters that were used to show numbers; it kept sizes clear by keeping position clear; moreover, it relieved the mind from thinking continually about the sizes of groups. The abacus may be described in another way. The abacus was developed out of man's efforts to deal with groups in a systematic way, and its use improved and made more fruitful man's efforts to handle groups systematically. So well was it contrived for a systematic arrangement of the groups with which man had to deal that it finally took over the necessity of giving continuous attention to the sizes of the groups and left the enumerator and computer comparatively free to give their attention to the manipulation of the counters. As the computer became more skilled in the manipulation of his counters, he gave less thought to the groups with which he was dealing and to their ordered relations. We have seen how the later development of the line abacus was in the direction of a more facile use of the counters through a reduction of the number of counters employed

and away from the former and reciprocal relations of the groups each to the other. We shall be reminded on a later chapter of the difficulty into which the computer was led through the employment of five and multiples of five as the base abacus when the necessity arose of multiplying his results of computation at the time when the Hindu-Arabian numerals were introduced in Western Europe.

Moreover, the neglect of the sense and substance of the groups by the medieval computer, which was made possible through the relief from the necessity of thinking about them which the abacus provided, may give some hint to the reason for the almost total neglect of the sense of our own position by teachers and textbooks today. We shall have occasion in later chapters to refer to the encouragement of such modern methods of teaching arithmetic as evidence of compensation to the neglect of the sense of our and the civilization of position. The psychology of the character of mathematics may be of some assistance in regulating the proceedings employed by the arithmetic teacher and arithmetic textbooks writers of our own day.

CHAPTER IV

ANCIENT NUMERALS

ARGUMENT

1. The first numeral was a set of counters that was left to stand as a record.

2. Constant use of a set of counters to stand for a group served to attract attention to the form in which the counters had been placed.

3. Reproducing the numeral became finally the act of copying a single form rather than one of representing all the individual details. Thus, a single numeral came to substitute for a set of counters in representing a group.

4. The use of numerals tended to pattern after the use of counters, and thus to become an aid in the development of number thinking.

5. In spite of the apparent mutual assistance of names, counters, and numerals, the arithmetic of today has experienced an extraordinarily retarded development.

6. The retardation was due to the inflexibility of numerals, to their fixity of form, and to their relative permanence when written as a record. A numeral system that was once useful in aiding number thinking could not change itself so as to keep pace with improved methods of number thinking.

7. The Roman system of numerals serves as an illustration:

- (a) Of irregularity in the use of a number base,
- (b) Of irregularity in representing large groups in relation to small related groups,
- (c) Of incompleteness as an aid in thinking,
- (d) Of the manner in which a numeral system may confuse, instead of keep clear and related, the ideas of size and number; in short,
- (e) Of a numeral system that did not employ the ideas of number relations that had been developed by other means.

I The Origin of Numerals

Following his creation of the art of recording, early man was content with his use of objects as an aid in recording phenomena. man acquired the motive to record with great degree of permanence the number ideas that he had gained and used under various circumstances. Whether the motive at the beginning was practical or logical, we have no means of determining. The fact that numerals were developed by various evidences man's interest in the recording of numbers alone.

The first numerals were not numerals at all as we know and use them today. They were in reality nothing more than relatively permanent impressions of such objects as man had used from earlier days. They are the marks of the time by serving as a record and in that way they served the original purpose of numerals.

Primitive man, as we have seen, learned to count by the use of all sorts of convenient objects as counters. Shells, stones, marks in the sand, marks in clay and the like. In him use marks in clay as counters was a definite step as when counting a group of objects, these *////* if not completely not erased, they remain as a record. They lack no other characteristics of permanency. They do not have identification about, and are not collected in bundles by reason of permanency. Such marks as counters *////* serve as a notation, a written record, of the number when used as counters. They thus possess, whether by accident or design, the character of a numeral. Although in themselves they possess a certain permanence, they undergo their first change of mind. After using and then abandoning such a set of counters as a record for a number idea, one would be expected to use the same kind of counters in the same way again to serve a similar purpose.

The fact that early numerals were simply counters used for the purpose of recording is evidenced by the following illustrations:

The Babylonian numerals 1 to 9 were formed somewhat as follows:

$\text{vvv} \text{vvv} \text{vvv} \text{vvv} \text{vvv} \text{vvv}$
 $\text{v} \text{vv} \text{vvv} \text{v} \text{vv} \text{vvv} \text{vvv} \text{vvv} \text{vvv}$
 $\text{v} \text{vv} \text{vvv}$

Chinese rod numerals from 1 to 5 are:

$\text{I} \quad \text{II} \quad \text{III} \quad \text{IIII} \quad \text{IIII}$

Roman numerals from 1 to 4 are:

$\text{I} \quad \text{II} \quad \text{III} \quad \text{IIII}^1$

II. THE ATTRACTIVENESS OF FORM

The arrangement of counters for the purpose of recording an idea of number gradually took upon itself a characteristic form. This form gradually impressed itself upon the mind, and came finally to stand out in clearer perspective than the individual counters that composed it. As a particular form of arrangement was used and observed again and again, it came to provide the means by which the idea it represented was recognized. One was able finally to recognize the numeral by its form, without having to count the objects that produced it. For example, the Babylonian numeral for nine

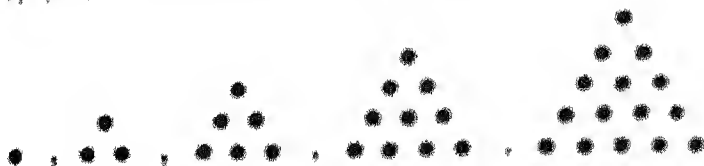
vvv has a characteristic form which is impressive. After a

short acquaintance with this numeral, one would be able to recognize it by its characteristic form.

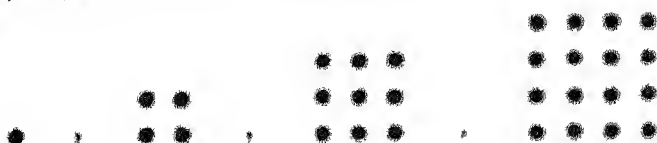
The Greeks were especially interested in the various configurations that could be taken by the objects they used originally as counters. They went to such lengths in their interest in form and arrangement as to classify certain numbers as *triangular*, *square*, *pentagonal*, and so forth. For example, triangular numbers are those that can be built up

¹ D. E. Smith. *History of Mathematics* II (Ginn and Company, Boston, 1925), pp. 37, 40-41.

as sums of the sequence 1, 2, 3, 4, 5, etc. Such numbers are 1, 3, 6, 10, 15, etc., and can thus be represented:



Square numbers are those that can be found up as sums of the sequence 1, 3, 5, 7, 9, etc. Such numbers are 1, 4, 9, 16, 25, etc., and can thus be represented:




Pentagonal numbers are those that can be found up as sums of the sequence, 1, 4, 7, 10, 13, etc.



Probably our word figure as referring to numbers and human 'figuring' can be traced to this source, entered in manuscript to Latin Europe by Boethius and for a thousand years regarded as a part of arithmetic on the eastern record of that age.

III THE HISTORICITY OF NUMBERS 3 AND 2

Because attention was attracted to the form of the present arrangement of counters that has been preserved as a record of the efforts of the one who was interested in getting along a record came to be directed more and more toward rather a perpetuation or a development of the form and was not so much toward a faithful representation of the numbers originally used to produce the form. The present arrangement 2 and 3 are merely current forms of the original CXX and CC . The Roman numeral X is said to have been called *decem* for ten.

¹ L. C. Karpinski *The History of Mathematics* (Chicago, Blaisdell and Company, Chicago, 1923), p. 18.

ment of ten counters.² One may go further on the same theory and hazard the guess that the Roman V was suggested by the following arrangement of five counters,  or by the angle made by the thumb and fingers of the open hand.

By whatever theories the origin of numeral forms may be explained, however, the fact remains that the forms developed more and more as conventionalized signs with less and less obvious relationship either to counters or to the number ideas they represented. The counters in a given arrangement produced a single form, and the singularity of the form, rather than the plurality of the counters, became the object of attention. The single form came to represent a single given group. Finally, it made little difference what character or sign was used to represent a group, so long as there were general agreement and understanding about the relation between the group and its sign. The laying of the rod numerals of the Chinese horizontally instead of vertically changed the values they represented from units to tens. Similarly, the counter and the numeral used by the Hittites for 'one' became, when written horizontally, the numeral for the group of ten. For example,  represented one;  represented ten. Thus the numeral lost its relation with counters and came to represent such number ideas as the social group chose to have it represent. The complete severance of the original relationship between numerals and counters is illustrated in the use of the letters of the alphabet as numerals. The Greeks used their first nine letters for units, the next nine for tens, and, adding three other characters making another nine, they used these for hundreds. The use of letters in the Roman system of notation is familiar.

The conventionality of the later numeral forms is to be seen in the difference between the method by which they were treated and the use of counters or of numerals that

² Smith, *op. cit.*, pp. 4, 67-68.

their numeral for eleven $\leftarrow v$. On the other hand, \overline{xx} was represented as twenty less one, $\leftarrow \overline{xx} \rightarrow v$, the symbol v , meaning 'less.' Further illustrations of the use of the additive and subtractive principles is to be found in the more familiar Roman numerals that will presently be described.

V. THE SERVICE OF NUMERALS

From the beginning, the written language of numbers has presented the possibility of rendering a double service. In the first place, the written numeral stands as a record of the results of thinking. After one has reached an objective, arrived at an idea of number, he may set down the appropriate numeral to record his result. As such, the numeral serves as a reminder. Because it thus frees the mind for further thinking, it is capable of performing its second and closely related service. As a record of our step in the process of thinking, it not only frees the mind so that it can go on to the next step, but it also serves as a check for looking up into a series of easy steps what otherwise would be a difficult performance. Like the language of any science, the language of number is both a means of expression and an instrument of thinking.

The service of numerals may be described in another way. The creation of counting and the invention of counting devices preceded the writing of numerals and demonstrated at the outset how the numerals were to be used. The invention of numerals in turn contributed to the oral language of number and to the employment of counting devices by permitting the computer to set down a partial result of his thinking and to proceed with an uninterrupted mind to the next step. Each instrument — number words, counters, and numerals — played its part in the thinking process, such as it was in the beginning, and each participated in mutual and reciprocal advantages. Number thinking had already progressed a long distance from its earliest beginning when the primitive mind first conceived of the written

The calculations now performed by children in the third and fourth grades required only a short time ago the services of a specialist. Until recent times calculation was a hidden mystery to the common man; it was a laborious and somewhat uncertain procedure to the expert.

One who reflects upon the history of reckoning up to the invention of the principle of position is struck by the paucity of achievement. This long period of five thousand years saw the fall and rise of many a civilization, each leaving behind it a heritage of literature, art, philosophy and religion. But what was the net achievement in the field of reckoning, the earliest art practiced by man? An inflexible mathematics so crude as to make progress well-nigh impossible, and a calculating device so limited in scope that even elementary calculations called for the services of an expert. And what is more, man used these devices for thousands of years without making a single worth-while improvement in the instrument, without contributing a single important idea to the system.¹

This criticism may sound severe, after all it is not fair to judge the achievements of a remote age by the standards of our own time of accelerated progress and feverish activity. Yet . . . with the slow growth of ideas during the . . . of reckoning presents a puzzling picture of desolate stagnation.²

The foregoing observations bring us face to face with a problem. In view of the fact that the use of numerical devices, developed in coördination with the use of numeral notation and the use of counters, making possible from their inception a higher level of thinking than could have been undertaken without them, and giving increased impetus to the development of number ideas, how can the long-continued and almost unbelievable stagnation in number thinking be accounted for? The advantages of a record as a substitute for memory and as an aid in thinking are evident. At the outset the use of numerals aided thinking. What accounts for the long halting of the progress once started?

¹ Dantzig, *op. cit.*, p. 29

The children may be put in another way. We look around us and observe many children in our schools, and find, progressing slowly and rapidly from crawling in the arrangement of groups in children, motorious, walking, and the more, and growing up in more and more and more abstract patterns. To us it appears that they move from the lower stages of crawling in the higher levels of the same, sometimes in a series of easy steps. They appear to be prepared by the stage previously taken for such progressing steps. Their earlier steps are in the field of the crawling and arrangement of objects in crawling, and appear to us very like the early steps the more look in crawling and in the manipulation of objects. We reflect that the children learn to crawl, later to walk and motorious, later to crawling and stands. We may pause to wonder why the more children we have after a long time in crawling and much progress a few weeks, some and motorious. We remember in the eye that some progress is made for some and that the child has the experience of the time as has been and may be guided around the children the more has made. But even with reflection give no explanation of the long and slow progress in crawling thinking. We can understand some small progress some, even some of many have been. But it is difficult for us, when we have the rapid strides made by some children, in understanding no progress at all for such long periods of actual experience.

VII THE INFLUENCE OF NATURE

We have the answer to our question in the fact that when children begin to be used after a given fashion they take upon themselves a habit of order and arrangement that becomes, with continued use, harder and harder to disturb. A habit may be altered and then it is gone, a set of routines may be set down in a given order, and these arrangements be easily changed, but when a reward is made it remains, if it does not pass, the quality of permanence. Let a set of materials be put together in a given order, and the order

resemblance, involved, and inadequate the early systems of number notation were as aids to thinking.

1. The Irregularity of Roman Numerals

The Roman system may be characterized as systematic only in its irregularity. It brings the user's attention back and forth from one number base to another. The base of ten is represented by the system and emphasized throughout, but the base of five is likewise used and is given almost as much emphasis. The ideas of ten, two tens, two hundreds, are fundamental in the system. V, I, M and those signs are consistently supplemented (and sometimes replaced) by the employment of the ideas of five, two tens, two hundreds. V, I, M. The system appears to work a definite advantage from the point of view of two secondary human conditions of ease. In some degree an advantage is concerned as we have seen in the use of five and appears in definite evidence in the modernized base system. On the other hand the shifting of attention back and forth between the base of ten and the base of five prevents the mind from taking the fullest possible advantage of either. A little effort is spared in the writing of the numerals, just as a little effort was spared in the "laying and setting" of numbers with the base objects, but the advantage gained is entirely null in the substance. The attention does not have a chance to settle upon one number idea, either of ten or of five, as a standard value around which all others may be grouped and by which all others may be judged.

Furthermore, irregularity characterized the method of representing larger groups as combinations of smaller groups. Although the method is predominantly addition, subtraction is used frequently enough to provide unusual definiteness. For example, thirty is recorded as ten and ten and ten, but sixty is recorded as ten added to fifty while forty is recorded as ten subtracted from fifty. XXX, L, XL. While there is apparent regularity in the addition and subtraction as

CHAPTER I

THE HINDI-ARABIC ALPHABET

Summary

1 The Hindi-Arabic numerals are used to illustrate the system proposed for numerals to represent groups larger than nine.

2 The numerals are to represent numbers. The names of groups are only represented by written words.

3 Capital, and perhaps others, represented with the method of representing one by writing the numeral in appropriate positions on the abacus.

4 The idea of the use of a numeral as one numeral that would keep the numeral to the numeral as the act of groups plus numeral gradually developed. At the time there were developed the idea of a sign for the single numeral as a means of holding position.

5 The use of a place holder suggested the Indian decimal system of numerals. Perhaps a sign of some numeral as 11 held position to show both numeral and also could now be employed without the use of the abacus.

6 The use of the same signified and characteristic sign in the Hindi-Arabic system.

7 The use from the appearance of having a numeral and the use appears to be the same as that of the numeral. The appearance, simplified with the change of the representation of position on the abacus, explains the close relationship of the Hindi-Arabic system to the modernized Chinese system in Western Europe.

8 Treating one as a numeral as elementary arithmetic today is comparable by much of the construction of children with the group one.

9 In elementary arithmetic the use is not a numeral but a place holder and should be treated as such.

我 們 了 解 到 該 項 工 程 的 總 價 為 1,200,000 元， 其 中 包 括 在 內 的 稅 金 為 120,000 元， 其 餘 1,080,000 元 為 淨 價。 該 項 工 程 的 總 價 為 1,200,000 元， 其 中 包 括 在 內 的 稅 金 為 120,000 元， 其 餘 1,080,000 元 為 淨 價。

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13. The non-filing of the 1980-81. 1981-82. 1982-83. 1983-84. 1984-85. 1985-86. 1986-87. 1987-88. 1988-89. 1989-90. 1990-91. 1991-92. 1992-93. 1993-94. 1994-95. 1995-96. 1996-97. 1997-98. 1998-99. 1999-00. 2000-01. 2001-02. 2002-03. 2003-04. 2004-05. 2005-06. 2006-07. 2007-08. 2008-09. 2009-10. 2010-11. 2011-12. 2012-13. 2013-14. 2014-15. 2015-16. 2016-17. 2017-18. 2018-19. 2019-20. 2020-21. 2021-22. 2022-23. 2023-24. 2024-25. 2025-26. 2026-27. 2027-28. 2028-29. 2029-30. 2030-31. 2031-32. 2032-33. 2033-34. 2034-35. 2035-36. 2036-37. 2037-38. 2038-39. 2039-40. 2040-41. 2041-42. 2042-43. 2043-44. 2044-45. 2045-46. 2046-47. 2047-48. 2048-49. 2049-50. 2050-51. 2051-52. 2052-53. 2053-54. 2054-55. 2055-56. 2056-57. 2057-58. 2058-59. 2059-60. 2060-61. 2061-62. 2062-63. 2063-64. 2064-65. 2065-66. 2066-67. 2067-68. 2068-69. 2069-70. 2070-71. 2071-72. 2072-73. 2073-74. 2074-75. 2075-76. 2076-77. 2077-78. 2078-79. 2079-80. 2080-81. 2081-82. 2082-83. 2083-84. 2084-85. 2085-86. 2086-87. 2087-88. 2088-89. 2089-90. 2090-91. 2091-92. 2092-93. 2093-94. 2094-95. 2095-96. 2096-97. 2097-98. 2098-99. 2099-00. 2100-01. 2101-02. 2102-03. 2103-04. 2104-05. 2105-06. 2106-07. 2107-08. 2108-09. 2109-10. 2110-11. 2111-12. 2112-13. 2113-14. 2114-15. 2115-16. 2116-17. 2117-18. 2118-19. 2119-20. 2120-21. 2121-22. 2122-23. 2123-24. 2124-25. 2125-26. 2126-27. 2127-28. 2128-29. 2129-30. 2130-31. 2131-32. 2132-33. 2133-34. 2134-35. 2135-36. 2136-37. 2137-38. 2138-39. 2139-40. 2140-41. 2141-42. 2142-43. 2143-44. 2144-45. 2145-46. 2146-47. 2147-48. 2148-49. 2149-50. 2150-51. 2151-52. 2152-53. 2153-54. 2154-55. 2155-56. 2156-57. 2157-58. 2158-59. 2159-60. 2160-61. 2161-62. 2162-63. 2163-64. 2164-65. 2165-66. 2166-67. 2167-68. 2168-69. 2169-70. 2170-71. 2171-72. 2172-73. 2173-74. 2174-75. 2175-76. 2176-77. 2177-78. 2178-79. 2179-80. 2180-81. 2181-82. 2182-83. 2183-84. 2184-85. 2185-86. 2186-87. 2187-88. 2188-89. 2189-90. 2190-91. 2191-92. 2192-93. 2193-94. 2194-95. 2195-96. 2196-97. 2197-98. 2198-99. 2199-00. 2200-01. 2201-02. 2202-03. 2203-04. 2204-05. 2205-06. 2206-07. 2207-08. 2208-09. 2209-10. 2210-11. 2211-12. 2212-13. 2213-14. 2214-15. 2215-16. 2216-17. 2217-18. 2218-19. 2219-20. 2220-21. 2221-22. 2222-23. 2223-24. 2224-25. 2225-26. 2226-27. 2227-28. 2228-29. 2229-30. 2230-31. 2231-32. 2232-33. 2233-34. 2234-35. 2235-36. 2236-37. 2237-38. 2238-39. 2239-40. 2240-41. 2241-42. 2242-43. 2243-44. 2244-45. 2245-46. 2246-47. 2247-48. 2248-49. 2249-50. 2250-51. 2251-52. 2252-53. 2253-54. 2254-55. 2255-56. 2256-57. 2257-58. 2258-59. 2259-60. 2260-61. 2261-62. 2262-63. 2263-64. 2264-65. 2265-66. 2266-67. 2267-68. 2268-69. 2269-70. 2270-71. 2271-72. 2272-73. 2273-74. 2274-75. 2275-76. 2276-77. 2277-78. 2278-79. 2279-80. 2280-81. 2281-82. 2282-83. 2283-84. 2284-85. 2285-86. 2286-87. 2287-88. 2288-89. 2289-90. 2290-91. 2291-92. 2292-93. 2293-94. 2294-95. 2295-96. 2296-97. 2297-98. 2298-99. 2299-00. 2300-01. 2301-02. 2302-03. 2303-04. 2304-05. 2305-06. 2306-07. 2307-08. 2308-09. 2309-10. 2310-11. 2311-12. 2312-13. 2313-14. 2314-15. 2315-16. 2316-17. 2317-18. 2318-19. 2319-20. 2320-21. 2321-22. 2322-23. 2323-24. 2324-25. 2325-26. 2326-27. 2327-28. 2328-29. 2329-30. 2330-31. 2331-32. 2332-33. 2333-34. 2334-35. 2335-36. 2336-37. 2337-38. 2338-39. 2339-40. 2340-41. 2341-42. 2342-43. 2343-44. 2344-45. 2345-46. 2346-47. 2347-48. 2348-49. 2349-50. 2350-51. 2351-52. 2352-53. 2353-54. 2354-55. 2355-56. 2356-57. 2357-58. 2358-59. 2359-60. 2360-61. 2361-62. 2362-63. 2363-64. 2364-65. 2365-66. 2366-67. 2367-68. 2368-69. 2369-70. 2370-71. 2371-72. 2372-73. 2373-74. 2374-75. 2375-76. 2376-77. 2377-78. 2378-79. 2379-80. 2380-81. 2381-82. 2382-83. 2383-84. 2384-85. 2385-86. 2386-87. 2387-88. 2388-89. 2389-90. 2390-91. 2391-92. 2392-93. 2393-94. 2394-95. 2395-96. 2396-97. 2397-98. 2398-99. 2399-00. 2400-01. 2401-02. 2402-03. 2403-04. 2404-05. 2405-06. 2406-07. 2407-08. 2408-09. 2409-10. 2410-11. 2411-12. 2412-13. 2413-14. 2414-15. 2415-16. 2416-17. 2417-18. 2418-19. 2419-20. 2420-21. 2421-22. 2422-23. 2423-24. 2424-25. 2425-26. 2426-27. 2427-28. 2428-29. 2429-30. 2430-31. 2431-32. 2432-33. 2433-34

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more, necessarily, than the smaller groups, except that one recognizes them as larger. But if the embryo arithmetician has no symbol that he can use to represent ten, or one hundred, he must give some special attention to these ideas. He must represent them in a different way. The difference in their representation helps to fix attention upon the special significance they possess. Wrapped up in the symbol X, for example, is the idea of *size*, but the idea has no special representation. Let the idea of the *size* of ten be expressed in a special manner, and the user will more likely give attention to the idea of *size*.

III. SIZE AND NUMBER

The Hindus in their oral language carried the expression of the decimal idea to great lengths. Working from the relation that ten bears to one, they proceeded by powers of ten up the scale to the ideas of hundred, thousand, and so on, and from these to the naming of the ideas by which the sands on the seashore, the raindrops of ten thousand years, and the "notes of the sun" could be numbered. They gave a special name to each power of ten, not only to the ones we name, such as tens, hundreds, and thousands, but also to the ones we express by compounding, such as ten thousand, hundred million, and so on. In their manner of expression they proceeded, as do we, from the higher powers in succession to the lower.

The method of number notation used by the Hindus followed their method of thinking and naming. Having a set of numerals to nine, they could proceed to write their ideas of ten and of the powers of ten. They proceeded from left to right in the writing of powers of ten from the higher powers in succession to the lower. Thus, the principle of position was maintained to the extent that order of arrangement provided for relative positions. In the early writing of their numerals to express ten and the higher powers of ten, the Hindus gave distinct expression both to the size of the

groups to be represented and to the number of the groups. They could either spell or abbreviate the names that indicated the sizes of the groups, and use the same numerals to indicate the numbers of groups.¹ Thus in writing the number which we express as 2154, they could write it, using their early numeral forms which corresponded, as 2 *sahas*, 1 *sata*, 5 *dasa*, 4; or as 2 *sa*, 1 *a*, 5 *d*, 4; or, if we abbreviate our own words for the sake of illustration, as 2h, 1s, 5d, 4. In writing the number that we express as 7653, they could write 7 *sahas*, 6 *dasa* (or, in our words, 7h, 6d).

IV. THE PRINCIPLE OF POSITION

The Hindu-Arabic numerals were used a long time before the use of position as an expression of value was suggested. For a long time the intellectual leaders were contented to make use of the numerals in one form or another to express the ideas of quantity that they developed by other means and continued to employ their counting devices for the computations they had to carry on. Our custom was that the thinking about number as powers of ten was carried on by the intellectual class but computations as a practical business procedure by the merchant class. Another and perhaps the chief reason was that the methods of computation in use were very highly developed and seemed to meet all their practical needs, while almost any other kind of notation was suitable to make a record of the results secured.

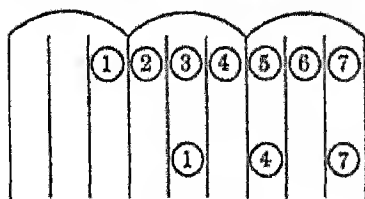
The Hindus had prepared themselves, however, as we have just seen, for the adoption of the use of position as a means of expressing values by their special development of the idea of size of groups. Now all they needed was to relate in their thinking the idea of size (expressed in numeration) with the idea of position (used in the placing of numerals).

¹ D. E. Smith and L. C. Karpinski, *The Hindu-Arabic Numerals* (Ginn and Company, Inc., 1911), pp. 40-41.

L. C. Karpinski, *The History of Arithmetic* (Ginn Educational Company, Chicago, 1923), pp. 29-30.

ers on the abacus), and they would be on the verge of an important discovery.

No doubt the use of the abacus suggested the relation. Numerals were used on the abacus before the advent of the Hindu-Arabic numerals in their present form and by persons who were unacquainted with the idea of place values of the numerals. The Romans at times used counters marked with their numerals for one to nine.² About the year 1000, Gerbert adopted the plan of numbering the counters for use on the abacus. His idea seems to have related more closely to the possible improvement of the abacus than to the possible improvement of his system of numerals. If, for example, he could put one counter marked "7" on a line instead of putting seven counters upon it, he would gain time in the placing of counters. The gain, however, was more apparent than real, because the computer



had to meet the difficulty of picking out the right counter each time.³ Because of the difficulty, Gerbert's abacus was not generally adopted.

Gerbert's abacus was devised somewhat as follows: The counters shown in the cut represent on the top row the number 1,234,567; those on the lower row, the number 10,407.

It would appear but a single step to the writing of numerals in various positions to represent values. Indeed, it appears that numerals were once so written on the so-called 'dust abacus.'⁴ In their proper positions, which were kept clear by the columns of the abacus, numerals were written in the dust of the counting board. The method was not generally adopted, and the method seems not to have trans-

² Karpinski, *op. cit.*, p. 26.

³ D. E. Smith, *History of Mathematics*, II (Ginn and Company, Boston, 1925), pp. 180-181.

⁴ Smith, *Ibid.*, p. 73.

ferred to the method of notation in use at the time. Though the method contained the germ of the idea that the system of numerals in use needed, it did not carry over to notation because it was employed as a possible improvement of the abacus as a counting device; when it was not generally adopted as part of the counting device, it dropped out of use and out of the minds of those who first employed it. This is one way of stating the case. Another method of statement is that the idea of position could not be developed in the numeral system so long as numerals were considered simply as accessories to the counting board in use, and not as a counting device possessing special merits in its own right.

V. THE EMPTY COLUMN

The reason why the expression of value by the use of the positions of numerals was so long delayed is that the numerals of the Hindus did not lend themselves to positional use. To be sure, they could be used on the dust abacus or on an abacus like Gerbert's, but when they were so used, it was the abacus and not the numerals themselves that kept position clear. The abacus possessed an advantage that the numerals could not have. The abacus was a mechanical contrivance that kept positions clear. The numerals did not.

Let us set the counters on the abacus to show three tens and two:

If now we substitute the numerals for the counters we can show the number thus:



In each case the machinery of the abacus keeps position clear. Let us express the number without the abacus, and

we have 3 2, which may be three hundred two; three thousand two, or thirty-two. When the columns of the abacus are filled, we have no such difficulty. We may express the number one thousand two hundred thirty-four in either of the following ways:

$$\begin{array}{|c|c|c|c|} \hline \text{—} & \text{=} & \text{≡} & \text{≡} \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \quad \text{or} \quad 1234.$$

In the expression, 1234, each numeral not only represents a number, but maintains its own position and helps to keep the rest in position. Owing, however, to the fact that one or another of the columns of the abacus is frequently empty of counters, one cannot use the nine numerals by themselves to keep each other in position. The empty column delayed the adoption of the principle of position.

1. The Sign of the Empty Column

If, in the expression of the number three thousand two, something can be placed between the symbols, 3 and 2, that will keep them in their proper positions, all will be well. Suppose we first have them in the contrivance that establishes their positions: $\begin{array}{|c|c|c|c|} \hline 3 & & & 2 \\ \hline \end{array}$ Let us now remove

the contrivance from all points except where it is needed and we have 3 \square 2, or as we now write it, 3002. Why did it require centuries for the discovery of this scheme? Why did it take so long to invent a symbol for the empty column, especially after symbols for occupied columns came to be used? The need for such a symbol is so obvious to us, and the connection between its use and the need for it seems so perfectly clear that we wonder how it is possible for any people who dealt in place values on the abacus to be so obtuse as not to make an almost immediate transition in their thinking from the empty column to a symbol that would represent it. The use of the zero is simplicity itself. If we wish to write three thousand two, we write 3 in thou-

sand's place, fill the blanks in hundred's and ten's places, and write 2 in unit's place.

If we will consider the mental attitude of earlier peoples toward the use of numerals, we may discover the answer to our questions. We may discover, too, a suggestion for an explanation of the difficulties and confusions that attend the learning of the zero in our schools today.

The numerals originated as representations of groups, or number ideas. They always served that purpose. There was no other reason for their existence. Anything as one, ten, or that was given a similar use, was accordingly considered to serve a similar purpose. For the representation of a given group, a numeral was used. No group at all needed no representation. Besides, since the numerals represented groups, they in no wise were related to the absence of a group. The absence of a group was represented well enough by the absence of a numeral.

So when numerals came to be used in place of the counters in the columns of the abacus and as representations of those counters, there arose no suggestion of the need of a sign for the absence of counters, for the empty column. When counters appeared, a sign was used, therefore, but no numeral, no sign appeared to be appropriate. There was no object in writing something to stand for nothing when the nothing was evident. Moreover, as has been pointed out, the abacus kept positions clear, and so there was no need of a sign for the empty column of the abacus so long as the abacus was used to keep either counters or numbers in place. The need for a sign for the empty column was very slow in suggesting itself, because there were no very evidences of no need at all to anyone who may have given thought to the matter.

If, under certain circumstances, necessary to the necessity of invention, such circumstances did not exist in connection with the invention of the zero. For all the purposes of ancient and medieval calculation, the abacus was sufficient.

ciently satisfactory. Arithmetic as a science had experienced very little development, and no demands for a better numeration issued from it. Ideas of exactness such as are in constant use today did not exist; in fact, they could not exist until they had been developed through the use of a better system of numeration than the one in use. The necessity for a more perfect system was absent, or, if not absent, was certainly not felt. Since the abacus served the purposes and needs of the calculators of the time, it is conceivable that every new use of the numerals that suggested itself was interpreted in terms of the abacus. As we have noted again and again, the abacus as a device for keeping position clear was entirely satisfactory and needed no additional aids.

2. The Importance of the Zero

However lacking may have been the practical need of a sign for the empty column at the time of its invention, someone did conceive the idea of such a sign and with the idea there developed the logical need for the sign. The sign completed the Hindu-Arabic system of numerals and made possible the use which we make of them today. Their use gradually superseded the use of counting devices, and thus paved the way for the progress of numeration from a mere means of expression to a means of thinking. With the numerals serving the double purpose, the calculator could use the numerals as an aid in thinking; and, as his thinking progressed, he could have at all points in his calculations a record of the steps he had taken. Serving thus the double purpose of thinking and recording, the numerals freed the mind at every step from the necessity of remembering the results of what had preceded, and at the same time they set the stage for progress to the next step.

In their discussion of the Hindu-Arabic numerals, Smith and Karpinski devote a chapter to the origin and history of the zero. They say:

[because of] the importance of such a sign, the fact that it is a prerequisite to a place-value system, and the further fact that without it the Hindu-Arabic numerals would never have dominated the computation systems of the western world.

They state further:

... If there was any invention for which the Hindus by all their philosophy and religion, were well fitted it was the invention of a symbol for zero. This marking of nothingness the crux of a transcendent achievement was a step in complete harmony with the genius of the Hindu.*

The following quotations from Dantzig's chapter discussing "The Empty Column" and from Smith's chapter on "The Psychology of Number" are pertinent:

... the achievement of the unknown Hindu, who some time in the first centuries of our era discovered the principle of position assumes the proportions of a world event. Not only did this principle constitute a radical departure in method, but we know now that without it no progress in arithmetic was possible. And yet the principle is so simple that today the dullest school boy has no difficulty in grasping it. In a measure, it is suggested by the very structure of our number language. Indeed, it would appear that the first attempt to translate the action of the counting board into the language of numerals ought to have consisted in the discovery of the principle of position.

Conceived in all probability as the spreader in an empty column on a counting board, the Indian empty was destined to become the turning-point in a development without which the progress of modern science, industry or commerce is inconceivable. And the influence of this great discovery was by no means confined to arithmetic. By giving the way to a generalized number concept, it played just as fundamental a rôle in practically every branch of mathematics. (1) The

* Smith and Karpinski, *op. cit.*, p. 14.

* *Ibid.*, p. 42.

* From Tobias Dantzig, *Number, The Language of Science*, p. 30. By permission of The Macmillan Company, publisher New York 1930.

history of culture the discovery of zero will always stand out as one of the greatest single achievements of the human race.⁸

While the abacus extended the usefulness of the number system and made possible elaborate calculations, it had the distinct disadvantage of being a thing quite apart from the mind that used it. The training of the mind which uses an abacus is not complete, because the processes of combination which such a mind uses are mechanical and external. The abacus served a useful purpose in that stage of civilization when the mind of man had not attained to a number system which is detached from all mechanical devices and yet possessed all of the virtues that the mechanical device contributes.⁹

The invention of the zero did not produce the principle of position; it merely provided for the appropriation of the principle by the written language of number. The abacus had always used the principle of position, but the abacus was a device outside of the individual — a mechanical device, not a language. By means of the zero it was made possible for man to take over the use of the principle of position and make it a part of his language of thinking and expression.

Let us state the importance of the zero in another way. The nine numerals of the Hindu-Arabic system were in use a long time before the zero was invented. The numerals could show number and hold position; they could not maintain positions, however, when the empty column existed in the case of a multitude of numbers that needed expression. An extra sign was needed to help the numerals to keep positions clear. The zero — the sign of the empty column — was invented, and it came into use as a device for filling up the spaces not occupied by the numerals.

⁸ Dantzig, *op. cit.*, p. 35.

⁹ From C. H. Judd. *Psychology of Social Institutions*, pp. 97-98. By permission of The Macmillan Company, publishers, New York, 1926.

VI. THE RECEPTION OF THE NUMERALS IN EUROPE

The Hindu-Arabic numerals were introduced into Western Europe in the twelfth century, and were generally adopted as the means of notation and calculation by the sixteenth century. "The fifteenth century saw the rise of the new symbolism; the sixteenth century saw it slowly gain the mastery; the seventeenth century saw it finally conquer the system that for two thousand years had dominated the arithmetic of business. Not a little of the success of the new plan was due to Luther's demand that all learning should go into the vernacular."¹

The Hindu-Arabic notation was very slow in making its way. The difficulty encountered was the confusion about the position into which the use of the new numerals had led the European calculators. We have referred in a preceding chapter to this abuse, which used the same to stand for the powers of ten and the spaces between the lines to stand for the half of such powers. The use of the spaces caused the true significance of the lines as exponents of the powers of ten to be obscured. For, when the new system, which was built solidly and entirely upon the arithmetic of representations — that is, of powers of ten — was introduced, the slight obscuring of the fundamental idea was sufficient to retard its adoption for the period of four centuries. We may get some notion of the confusion that existed by noting the following explanation taken from an early English book:

Every of these figures hathen byen sette & so named, as the stands in the first place of the counte.

If it stande in the seconde place of the counte, he hathen ten tymes hym selfe, as the figure 2 have 20 when he stande in the seconde place, that is twenty, for he hathen selfe hathen ten, & ten tymes twene is twenty. And for he stande in the thirde place & in the seconde place, he hathen ten tymes hym selfe, and so go forth . . .

¹ Smith and Karpinski, *op. cit.*, pp. 149-150.

² *Facit*, p. 145b.

When the Hindu-Arabic numerals were introduced into Western Europe, they were distinguished in the minds of most users by their most characteristic feature. No other system of numerals employed a zero. The zero was an interesting and striking peculiarity of the new system. A name for the zero came to be used to describe the use of the whole system: to *cipher* meant to use the system which had a *cipher* as a symbol. The masses judged the new system by appearances, and so they thought of the zero as a numeral; that is, as a sign for a number. It appeared to them that the zero was used like the other signs of the system; hence, it must be a numeral, standing for nothing.

VII. THE ZERO IS A PLACE HOLDER, NOT A NUMERAL

The zero is the characteristic feature of the Hindu-Arabic system, because this system depends not merely upon numerals to show quantity, but also upon the positions in which the numerals are written. In this system the size of a group is indicated by position, and the number of the group by the numeral written in that position. Each numeral thus serves the double purpose of showing a number and of keeping the rest of the numerals in their proper places; the 2 in 521 shows 2 *tens* and puts the 5 in *hundred's* place. Suppose one has nothing to write. If that is all the purpose he has in mind, he writes nothing. There is no economy or common sense in writing something — the zero — for nothing. But suppose one has nothing to write in a given position, yet needs some means of keeping the numerals he does have to write in their proper places — for example, in writing *five hundred twenty*. One has no units to write. Why write zero for nothing? There is no need to indicate nothing by a sign, but there is need to put the 2 and the 5 in their proper places. The zero in 520 serves that important and essential purpose. One never needs the zero until he has to write the quantity ten, or certain quantities larger than ten. He needs it then to hold position. The zero is a place holder.

It is interesting to note that the original term *ben zero* did not mean a group of none. Both the Hindu term *benzero* and the Arabic term *sifr* meant empty, or void. The sign, 0, was originally a sign for the empty reckoning. It was left to the masses, who watched the use of the zero and who noticed only what their eyes could see, to convert the original and useful meaning of zero to nothing. There grew up a difference between the arithmetical (actual) use of the zero and the popular (apparent) use of the zero.

Teachers of elementary arithmetic might do well to take note of this difference, because failure to show just the distinction is responsible for the confusion and lack of competence in employing the zero in computation. Children who are engaged in learning arithmetic must actually employ the zero for what it is intended and for what it is taken on as a place holder. They will, of course, be confused and misled by appearances, since the use of the zero looks like the use of a numeral. Let the teacher, however, lead to make the necessary distinction, or, what is worse, attempt to teach the children the apparent use of the zero as a symbol for nothing while neglecting entirely to call attention to the actual use of the symbol, and the confusion of the children will be worse confounded. As a remedy, the teacher often turns to better, more complete, and more skilled methods of teaching the apparent use of the zero, and the teacher just about as often finds himself in an almost hopeless situation about the failure of improved methods of teaching apparent uses as the children are in their attempts to convert apparent uses into actual uses.

In fairness, however, to the good intentions of teachers it must be admitted that the efforts to improve these methods of teaching the apparent uses of the zero are not without some measure of success. It often appears to teachers that the children will have to add, subtract, multiply, and divide the zero just as they add, subtract, multiply, and divide the nine numerals are used. So, as a means of preparation,

the 'zero combinations' are 'taught.' There is much explaining and more drills. Through persistent effort, children finally learn to set down the approved answers when the zero is used. Since the actual use and meaning of the zero are neglected, what the children learn to do is necessarily barren of meaning. Their human minds are turned into machines for use in responding to the 'zero combinations' by giving what the teacher says are the correct answers. Like machines they learn to respond, but like machines they fail to think and to understand. Moreover, since the actual positional use of the zero is neglected in teaching the apparent uses, the children who finally learn how to respond with correct answers are not helped to keep the positions of the answers clear. If they learn about the importance of position, they learn it incidentally or as a separate and unrelated item.

VIII. THE ZERO IN LATER MATHEMATICS

The reader of the foregoing statements that the zero is not a numeral, standing for nothing, may recall the frequent use of the zero as a numeral in later mathematics. The zero is used in later mathematics as a numeral, as a position on a scale, and as a division point between positive and negative numbers; and its use for these purposes is the same as the use of the nine numerals. All such uses, however, belong in the mathematics that has been developed out of *elementary* arithmetic. In the system of numeration that deals with the sizes and numbers of groups and with the various arrangements to be made of groups, the zero is unlike the numerals and has no use as a representation of no group, or nothing. Because it does not carry the idea of no group, it may not be classed with the numerals that stand for quantities, though its use may appear the same as the uses of the numerals. If the reader will reflect upon the uses the zero finally takes on in later mathematics, he will note that these are quite unlike any that are required in the

beginnings of arithmetic. Children of the beginning grades do not study points on a scale; they study groups. They do not deal with the relations between positive and negative numbers; they deal with the arrangements of groups. They do not study the exceptions to the general law of combination that the zero as a numeral, and as representative of nothing, makes necessary; they study how groups may be taken apart and how they may be combined. The groups studied by beginners are actual groups of objects. 'nothing,' as a quantity, is an abstraction too far removed from the world of actuality for their conception.

IX. THE MERITS OF THE HINDU-ARABIC NUMERALS

The Hindu-Arabic notation is a superior system because of its simplicity. It uses separate symbols to stand for those number ideas that may be learned and handled as distinct ideas; it uses the symbols to show the relations by which one may conceive and handle those ideas that are not separate and distinct; it sharply distinguishes the idea of sum and the idea of number; it uses position to represent powers of ten; it makes possible, and also strongly suggests, the principle of dealing with tens just as one deals with units and it permits one to make use of all these advantages with a minimum of conscious effort by keeping all relations clear and the system. The merits of this system have been indicated in our previous discussions, we shall have occasion to refer to them again and again in later discussions. It will be sufficient at this point of our well rounded argument that the Hindu-Arabic system finally appropriated to itself all the advantages of the principle of position that had developed in the use of counting devices and made these advantages easily accessible to human minds by integrating them into the written language of numbers. No longer did the calculator have to chronicle them in himself on the counting board; he now pronounced them as part of the language of thinking.

CHAPTER VI

FRACTIONS

ARGUMENT

1. Ideas of parts and ideas of groups developed somewhat similarly.

2. The first fraction was a broken part of a thing, and it was thought of as a single part, or unit fraction.

3. The idea of the unit fraction was one relating to size. The idea of number was cared for automatically.

4. Sexagesimal fractions — sixtieths — cared for the idea of size automatically. Thinking with such fractions related to number of parts.

5. Uncial fractions — twelfths — were similar in character to sexagesimal fractions. In their use, however, attention was shifted back and forth between ideas of number and ideas of size. No certain way was provided for uniting the two sets of ideas in thought.

6. The use of fractions was uncertain, inexact, and confusing so long as size was considered to the neglect of number, as in unit fractions, or number was considered to the neglect of size, as in sexagesimals and uncials.

7. The Hindus combined the two sets of ideas. They provided the present way of representing both ideas at once.

8. Quite by accident, the decimal system of notation was appropriated for the representation of fractions. In the writing of decimals the numerals are written to show the numbers of parts, and their positions are used to show the sizes of parts.

9. The idea of percent is the idea of the hundredth part. It develops through the special study of hundredths.

10. The meaning of percent is clarified through consideration of the original meaning of "per centum" and of the derivation of the percent sign.

11. The idea of the fraction finds much practical use as a

means of stating the comparison between numbers that is of indicating one number in its relation to another number that may be more familiar.

12. Such practical use is in connection with one or number of the so-called 'three kinds of problems'

- (a) Finding the part, or percent, of a number.
- (b) Finding what part, or percent, one number is of another number;
- (c) Finding a number when a part, or a percent, of it is known.

13. Acquiring familiarity with the three kinds of problems is in the main the process of learning the distinctions between them and of looking for such distinctions.

I. PARALLELISMS BETWEEN IDEAS OF GROUPS AND IDEAS OF PARTS

At many points in their development ideas of groups and ideas of parts moved closely in parallel both back and ancient history. Both had their languages of geographical experience. Both were marked at the center both number went processes of refining and classification that, in respect, seem extraordinarily close both experienced through long centuries the solidifying influence of an unchanging written language. Finally, both appropriated as a common language the decimal system of notation, which makes possible an easy transference of thought from one end of a scale to the other.

As the history of numbers brings into relief the distinctive features of ideas of groups, so the history of numbers comes off in clear perspective the essential characteristics of ideas of parts. The present chapter is introduced therefore to present a brief sketch of the development of those ideas that are commonly called 'fractions' with the thought of uncovering clues to explanations of the failures and successes of pupils in attacking this difficult chapter in their study of arithmetic.

II. WHAT FRACTIONS ARE

Fractions are parts. The word 'fraction' derives from the Latin *frangere*, "to break." A part, or a fraction, of a thing was originally "broken off" the thing. Thus, pieces of bread were broken off the loaf just as a piece of candy is now broken off the stick and as a part of an apple is broken off when the apple is broken in two. In Medieval Europe, fractions were called "broken numbers," and sometimes "brokens." A fraction of a fraction, such as $\frac{1}{2}$ of $\frac{1}{2}$ was thus designated a "broken of broken." The Romans called their sixtieths "little parts." Their division into sixtieths resulted in their *partes minutiae primae*, "first little parts"; and their second division into sixtieths of sixtieths resulted in their *partes minutiae secundae*, "second little parts." From the two phrases come our familiar words 'minutes' and 'seconds.'

III. UNIT FRACTIONS

1. The First Fractions

The first fraction to which early peoples gave their attention was of the type that we now designate as a unit fraction; that is, "one part," though the term is now the carrier of more meaning than the original idea of part possessed. The first fraction was merely the broken off piece of a thing — a part of it, whether large or small. Size was, of course, distinguished from the beginning, though attention to size was but crudely comparative. The type of attention given and the type of idea resulting were of the sort manifested by the small child who distinguishes between a stick of candy and a piece of the stick or between the big and little pieces when the stick is broken in two. Though the older child and the adult may recognize the larger piece as two-thirds, say, of the original, each commonly regards it, not as two pieces, each the size of the smaller, but as what it actually is: namely, one piece. Thus did early man. Having at the

outset no system of thinking to bring to four equal halves parts, he regarded only the actualities. A part was a part, whether large or small, and each part was a unity.






As early man developed the art of counting, he was invested with a means of enlarging the meaning of his original idea of the part in the direction of what the term 'unit fraction' (a fraction of which the numerator is one and the denominator is any integer) now means to us. Counting provided a set of number names that enabled early man to keep in mind two or more parts into which a thing may have been divided. This set of names, combined with a recognition of the equality of sizes resulting from various methods of division, made possible the creation of the idea of halves, thirds, fourths, etc. Thus, when a skin, as a collection of arrowheads, or any other item of property, was divided into two equal parts, and attention was given to one part as the equal of the other, the idea of the half was born, or, when the number of moons (months) during the rainy season was counted and the three were regarded as of equal duration, the idea of the third developed. Attention to equality of sizes and the possession of a number name, as a means of fixing attention upon the relation of each to the whole, created the idea of the unit fraction: the half, the third, the fourth -- by whatever name each was called.

2. Writing Unit Fractions

The correspondence between the unit-fraction idea and the number name that helped to fix its meaning was expressed in the earliest written records. All that was needed was a sign to indicate the general idea of a part. This sign, when then he joined with the appropriate numeral to indicate the size of the particular part to be represented. Such a method of notation was sufficient for primitive needs. First came thought of one part at a time; parts needed to be written one part at a time. Illustration may be drawn from the fractional notation of the early Egyptians.

The sign for a fraction used by the Egyptians was a crude ellipse. This sign indicated that a part was being shown. Just below the sign the numeral (sign) that indicated the size of the part was written. The illustration shows in the left column certain Egyptian numerals and in the right column the corresponding fractional expressions.


The only fraction not a unit fraction that the Egyptians could write was *two-thirds*. They apparently understood

/// three	 one third
//// five	 one fifth
∩ ten	 one tenth
∩∩∩ thirty	 one thirtieth
∩∩∩∩ forty-two	 one forty-second

this fraction, and for it they used the special sign Φ . All other fractions were thought of each as a *single part*, or unit fraction.

The method of writing unit fractions accomplished two results. First, it freed the mind for a more detailed and exact consideration of parts that were not equal divisions of the whole; second, it limited the consideration of such parts to their relations with unit fractions. When an Egyptian had eaten a *part* of his cake, for example, he had a *part* of it left, and he thought of this also as *one part*. Let us say that he had eaten a fourth of it. He now had this *one part* remaining.



He could write  to show how much he had eaten, but he had no way of showing by a single sign that *three-fourths* was left, because the part left was actually *one part*. There

was no method of thinking *three parts* as a *single part* than was too difficult an abstraction. True, the *single part* needed be divided into three equal parts as the discussion involved the carried through as a matter of thought. In such cases three parts resulted, the three parts were *separate parts*, and there had to be thought of, and written, as much *one* as *two* there is, as three unit fractions. Actually, the division was carried through, or thought through, so as to produce the smallest possible number and the largest possible name of equal divisions. The *three-fourths* part was divided thus:



with one half and one fourth remaining. In a similar manner, a part the size of *five-eighths* was represented by a part the size of *seven-eighths* was represented by a part the size of *nine-eighths* was represented by a part the size of *eleven-eighths* was represented by a part the size of *thirteen-eighths* was represented by a part the size of *fifteen-eighths* was represented by a part the size of *seventeen-eighths* was represented by a part the size of *nineteen-eighths* was represented by a part the size of *twenty-one-eighths* was represented by a part the size of *twenty-three-eighths* was represented by a part the size of *twenty-five-eighths* was represented by a part the size of *twenty-seven-eighths* was represented by a part the size of *twenty-nine-eighths* was represented by a part the size of *thirty-one-eighths* was represented by a part the size of *thirty-three-eighths* was represented by a part the size of *thirty-five-eighths* was represented by a part the size of *thirty-seven-eighths* was represented by a part the size of *thirty-nine-eighths* was represented by a part the size of *forty-one-eighths* was represented by a part the size of *forty-three-eighths* was represented by a part the size of *forty-five-eighths* was represented by a part the size of *forty-seven-eighths* was represented by a part the size of *forty-nine-eighths* was represented by a part the size of *fifty-one-eighths* was represented by a part the size of *fifty-three-eighths* was represented by a part the size of *fifty-five-eighths* was represented by a part the size of *fifty-seven-eighths* was represented by a part the size of *fifty-nine-eighths* was represented by a part the size of *sixty-one-eighths* was represented by a part the size of *sixty-three-eighths* was represented by a part the size of *sixty-five-eighths* was represented by a part the size of *sixty-seven-eighths* was represented by a part the size of *sixty-nine-eighths* was represented by a part the size of *seventy-one-eighths* was represented by a part the size of *seventy-three-eighths* was represented by a part the size of *seventy-five-eighths* was represented by a part the size of *seventy-seven-eighths* was represented by a part the size of *seventy-nine-eighths* was represented by a part the size of *eighty-one-eighths* was represented by a part the size of *eighty-three-eighths* was represented by a part the size of *eighty-five-eighths* was represented by a part the size of *eighty-seven-eighths* was represented by a part the size of *eighty-nine-eighths* was represented by a part the size of *ninety-one-eighths* was represented by a part the size of *ninety-three-eighths* was represented by a part the size of *ninety-five-eighths* was represented by a part the size of *ninety-seven-eighths* was represented by a part the size of *ninety-nine-eighths* was represented by a part the size of *one*.

Computations with unit fractions were no less readily seen, matters of extraordinary difficulty. The reason was that the idea of the unit fraction was exclusively an idea of one of a single part, and did not admit any other way of thought of number of parts other than the unadorned thought of *one*. Thought of number of parts required no other idea of *one* as the number under consideration. More than a *single part* could not be written as a *single* unit, *seventy-nine* was written

¹ The clearest historical treatments of these and corresponding topics are to be found in the chapters on arithmetic and fractions in the following:
D. E. Smith. *History of Mathematics*. 11. McGraw-Hill Book Co. (New York, 1925.)

L. C. Karpinski. *The History of Arithmetic*. (Chicago, 1925.)

Vera Sanford. *A Short History of Mathematics*. (Boston, 1925.)

not be carried in thought as a single part. The unit fraction concentrated upon *size*, but neglected *number*.

IV. SEXAGESIMAL FRACTIONS

The difficulties involved in the use of unit fractions led early peoples to the invention of a method of thinking and writing parts which in its essential feature was the extreme opposite of its predecessor. This was the method of the Babylonian and Greek astronomers of reducing their fractions to sixtieths and sixtieths of sixtieths. The first of these reductions, as indicated in a foregoing topic, gave us our *minutes*, and the second one of them, our *seconds*. This was the method of seeming to ignore *size* — that is, of so standardizing sizes that they could be readily carried in thought — and of concentrating upon *number* of parts.

Apparently, the use of unit fractions led to the conviction that a convenient method of dealing with number of parts was needed; and, apparently, the conclusion seemed to have been made that, since it was so extremely difficult to think of *number* when *size* of parts was represented, it would be equally difficult to think of *size* when *number* of parts was represented. The difficulty seemed to lie in the demand that the two essential features of the fraction, *size* and *number*, be thought of at once and each in relation to the other. At any rate, early peoples came finally to realize that attention needed to be given to number of parts, and when they did, they tried to dispense with *size*. They worked the size of any given part into so many 'minutes,' or so many 'seconds,' or both (not alone of the hour and degree, but of anything else); and then they proceeded to forget about parts and sizes in concentrating upon number of minutes and number of seconds just as they attended to whole numbers. Just as we do not have to think of 10 minutes, or 45 minutes, as parts of an hour, so early peoples were not compelled to think of parts and the sizes of parts when they used sexagesimals. Though they were actually parts,

sexagesimals could be dealt with in computations as whole numbers were dealt with.

After the invention of sexagesimals, early progress continued to use their unit fractions for ordinary purposes. They continued to think and write the *half third fourth* and so on; but they could not use such parts as *four-fifths* and *five-sixths* as we do today. So when they had to think of parts of such sizes, they reduced them to sexagesimals. Instead of *four-fifths*, they thought of 48 minutes, and instead of *five-sixths* they thought of 50 minutes and 10 seconds, or 37½ minutes. Today we do not usually speak of *two-thirds* of an hour, but of 40 minutes. Today we seldom write ½ of an hour, but 30 minutes and 10 seconds.

V. ROMAN UNITS

Although sexagesimals lent themselves readily to the exact computations of the astronomers, they were chosen little for common usage. The sixtieth part of the common range of life is too small a division for practical purposes. For example, there is no point in the measuring of broths and of cloth that are bought and sold as sixtieth fractions of the appropriate units of measure. Everyday usage requires a division into larger parts; that is, into parts with smaller denominators.

The Roman *uncia* was the result of such a division. The *uncia* was the twelfth part of the Roman *sextans* and also the twelfth part of the Roman pound, and in the measure of our English "inches" and "ounces." Likewise it is necessary to keep in mind, the *uncia* was the twelfth part of *capitulum*. Here was a division, possessing the advantages of sexagesimals, small enough for convenience, and yet large enough to be readily perceptible. These advantages, however, were not put to use, and the common understanding of fractions was not greatly advanced. The reason was that the Romans continued to use their *sexagesimal* with their *uncia* in thinking of parts, and were consequently

influenced the more by the limitations of both systems than by the advantages. Instead of learning to think both *size* and *number* together into single ideas of parts, they continued to attempt to forget number when considering size, and to forget size when considering number. All this seems to be indicated in the way they mixed the two contrasting sets of ideas in thinking, naming, and writing the fractions in common use. Below are given some commonly used fractions, their names, what the names meant, their symbols, and what the symbols showed.²

<i>The Part</i>	<i>Its Name</i>	<i>Meaning</i>	<i>Its Symbol</i>	<i>What the Symbol Showed</i>
$\frac{1}{12}$	<i>uncia</i>	twelfth	—	one twelfth
$\frac{1}{6}$	<i>sextans</i>	sixth	=	two twelfths
$\frac{1}{4}$	<i>quadrans</i>	fourth	= —	three twelfths
$\frac{1}{3}$	<i>triens</i>	third	= =	four twelfths
$\frac{5}{12}$	<i>quincunx</i> (<i>quinque unciae</i>)	five twelfths	= = —	five twelfths
$\frac{1}{2}$	<i>semis</i>	half	S	one half
$\frac{7}{12}$	<i>septunx</i> (<i>septem unciae</i>)	seven twelfths	S —	{ one half and one twelfth
$\frac{2}{3}$	<i>bes</i> (<i>bi as</i>)	two parts (two thirds)	S =	{ one half and two twelfths
$\frac{3}{4}$	<i>dodrans</i> (<i>de quadrans</i>)	one fourth away from one	S = —	{ one half and three twelfths
$\frac{5}{6}$	<i>sexrans</i> (<i>de sextans</i>)	one sixth away from one	S = =	{ one half and four twelfths
$\frac{11}{12}$	<i>deunx</i> (<i>de uncia</i>)	one twelfth away from one	S = = —	{ one half and five twelfths

The confusion here in method of thinking is evident. Notice the last fraction, which we would speak, think, and write simply as *eleven-twelfths*. We would imagine that the Romans might have thought of it as *eleven unciae*; but instead, they thought of it as a unit fraction, namely, one *deunx*. Then, when they wrote it, they had to think very differently about it, because they wrote it as "a half and

² Smith, *op. cit.*, pp. 208-209.

five-twelfths." The way they showed their thinking may be illustrated by reference to other fractions. Thus, a notional part was called *sextans sextilis*, "a twenty-fourth and a forty-eighth"; an eighth part was called *sextans octavus*, "a twelfth and a twenty-fourth", a three-sixteenth part was called *sextans sestas*, "a sixth and a forty-eighth", and a fifteen-sixteenth part was called *sextans undecima*, "an eleventh away and a forty-eighth."

VI. THE PRESENT METHOD OF WRITING FRACTIONS

Slowly and gradually it was learned that there was trouble in trying to forget the number of parts by leaving out the numerator in unit fractions, or in trying to forget the size of parts by leaving out the denominator in compound unit and mixed. If 'minuten' were 'minuthe' and 'sextans' were 'twelfths,' and if sixtieths and fortieths may both be shown by the writing of numerals with the sign for unit fractions, why not use numerals to show these and other mixed cases, more than one part has to be shown? After a long, long time someone must have asked the question. "Why not a double use of numerals at one and the same time to show both size and number, which heretofore have been shown separately?"

The Hindus seem to have been the first to discover a way of using numerals to show both size and number of parts at once. More than a thousand years ago the Hindustani began to show the two by writing the figure for the one part under the figure for the other. At first, they combined the two. Later, they used it: $\frac{1}{2}$. In early printing, the bar was often omitted, because of the difficulty of writing it up to 1999. Because of trouble in keeping the lines of type straight, denominators were often spelled, and numerators, sometimes, were shown by the numerals on their sides. $\frac{2}{3}$ for $\frac{2}{3}$, for example.

Since the Roman numerals were used as figures up to the four or five centuries ago, the Europeans, when they began

to use the Hindu-Arabic numerals, and gradually to appropriate the Hindu form of the written fraction, mixed the two numeral systems. Thus, in a *book on arithmetic*, printed in 1514, fractions were printed in Roman numerals arranged in the form of the Hindu fraction:

$$\frac{\text{I} \quad \text{I} \quad \text{II} \quad \text{IX} \quad \text{XX}}{\text{III} \quad \text{V} \quad \text{VI} \quad \text{XI} \quad \text{XXXI}}$$

VII. DECIMALS

The idea of using decimals was another seemingly simple idea that actually was a long time developing. Some notion of the idea came from the use of sexagesimals. Like decimals, sexagesimals have a common base. Moreover, in writing sexagesimals, the parts are written in order of size from left to right, and, thus, the idea of the use of position is not absent. In preceding chapters, we have noted how the idea of writing tens, tens of tens, and so on, in different positions slowly led to the writing of the Hindu-Arabic numerals in different position to show different sizes of groups. Logically it was but a step, and a closely related one, to the thinking of tenths, tenths of tenths, and so on, and to the writing of these parts in different positions according to respective sizes; but actually it was a long time before the step was even thought of.

The idea of the decimal seems to have come pretty much by accident. The Hindus stumbled upon the idea almost a thousand years ago in their roundabout method of extracting the square root. They discovered that if a number is not a perfect square, its root might be approximated by finding the root of the number with an even number of zeros added and then dividing the root by 10, 100, 1000, and so on, according to whether the number of zeros added was two, four, six, and so on. Thus, to extract the square root of 2, six zeros were added. The root obtained, 1414, was then divided by 1000, giving $1\frac{414}{1000}$. The germ of the idea was also developed in connection with a method of dividing numbers by 10, 100, 1000, and so on.

today, though he did not make use of the decimal point. In writing whole numbers and decimals, he used figures in circles to indicate positional values: ⑥ showed the place to be units; ① showed the place to be tenths; ② showed the place to be hundredths; and so on. Thus, to show 42.875, one or the other of the following forms was used: $\overset{\textcircled{0}}{42} \overset{\textcircled{1}}{8} \overset{\textcircled{2}}{7} \overset{\textcircled{3}}{5}$, or $42\textcircled{0} \ 8\textcircled{1} \ 7\textcircled{2} \ 5\textcircled{3}$.

After Stevin's time, as people began to develop the idea of decimals, they developed less confusing ways of writing them. Sometimes they used a comma between the whole number and the decimal; finally, they learned to use a point between the two. Thus, two and five-tenths was once written 2,5. Often the comma was used just to separate the whole number and the decimal, and other signs were used to indicate the fractions. Thus, in 1616, a number like 2.758 was written 2,7'5''8''' . Sometimes, when the comma was used, the writer would seem to forget the decimal form of notation, and would revert to the form of the common fraction. Thus, instead of writing 65.5, or writing it with a comma, 65,5, forms like these were used: $65\frac{1}{2}$ and $65\frac{5}{10}$. At times, the size of the fraction was indicated by numerals written under the numerator of the decimal. Thus, in a book written in 1685, the fraction .00438 was written $100\frac{438}{100000}$. At other times, other signs for the decimals were used. Thus, in a book written in 1657, numbers which we write as 2.5; 6.75; and 14.085 were written as 25...(1); 675...(2); and 14085...(3). In parts of Europe today, the decimal point is placed in the middle of the line: 3·75.

VIII. THE IMPORTANCE OF POSITION

The same rules apply in the writing of decimals as apply in the writing of whole numbers. In the writing of whole numbers, the *number* of ones or of groups is shown by the numerals, and the *sizes* of the groups, whether ones, tens, hundreds, or larger, are shown by the positions in which the

numerals are written. In the writing of denumerals the number of the parts to be represented is shown by the numerals and the size of the parts are shown by the position in which the numerals are written.

IX. Size and Number

The foregoing paragraphs may be summarized as the statement that the fraction possesses two distinguishing features; namely, size of parts and number of parts, and that fruitful consideration of the fraction requires the study of both. Later discussion will indicate that the pupil frequently meets with difficulty in dealing with the denominators of fractions. Such difficulty is with the denumerals, both historically and in the requirements of the pupil as he first strikes the attention. But however teachers and text books either take an understanding of our first paragraph or neglect to stress the importance of size or consider the pupil in his consideration of size, much confusion results.

The pupil comes to the study of fractions with an anxiety to deal with number. Perhaps, therefore, the understanding of number in the study of parts may be taken for granted. But not so the feature of size. He cannot be brought to think exactly as regards this feature. Very often our first pupil is misled in his consideration of size by confusion, discussion forced upon him by teachers and textbooks. It is the common practice to advance such definitions as the following: "The denominator shows the number of parts into which the thing has been divided." The denominator shows the number of parts to be taken. In short, both the denominator and numerator are used to show number of parts which is only partly true; and then, an error in reasoning is committed where none has been made a misunderstanding arises like "to be taken," or "that you have" etc. To avoid such definitions, the teacher understands why the pupil who at the outset had no trouble distinguishing between size and

fourths, etc., proceeds to add two-thirds and three-fourths as follows:

$$\frac{2}{3} + \frac{3}{4} = \frac{5}{7}$$

The pupil, having been confused about *sizes*, or having had no systematic guidance in the recognition of *sizes*, or having been taught to treat them as *numbers*, knows very well that he can add 2 parts and 3 parts, and the result will be 5 parts. He proceeds so to add, as just indicated. Since the denominators in the addends possess a false meaning, he might just as well set down a false meaning (or any meaning) in the denominator of his answer.

The history of fractions reveals that the race made little, if any, progress in the development of ideas of fractions so long as size was emphasized at the expense of number, or so long as number was emphasized at the expense of size. The same revelation is to be noted in the case histories of pupils in the school. They make no progress in developing a working understanding of fractions so long as either distinguishing feature of the fraction is neglected in their thinking. *Pupils must be led to study size as well as number.*

X. PERCENTS

1. The Idea of Percent

The idea of percent began to grow in the minds of people a long time before they began to think about the value and use of decimals. The idea was developed and used in connection with the levying of taxes, the grouping of soldiers, the charging of interest, and the computation of rates of gain or loss in trading.

The Romans levied some of their taxes in hundredths of the value of the property taxed, and they levied other taxes in parts that are closely related to hundredths. For example, when a slave purchased his freedom, a tax of $\frac{1}{10}$ of the purchase price was collected by the government; when a slave

was sold, $\frac{1}{2}$ of the price received was reckoned good when goods were sold at auction, a law of $\frac{1}{2}$ was reckoned.

In the Roman army a company was composed of 100 soldiers. The company was called a *centuria* a century. Thus, a small number of soldiers were at every part of the century that is so many hundredths.

When a small value was needed at the time, frequently, the value of small sale of less value than gold was not required. It was then, for example, in those times, that the value of the dollar of money was reckoned as a hundredth.

In Asia, the value was reckoned as many cents for each one value hundred, and we find in the writings of 600 years ago such expressions as *III p c*, *II p c*, and *VI p c*, which meant 30 out of a hundred, 20 out of a hundred, and 6 out of a hundred. The letter *p* stood for "per," which means "by" or, as we use it, "out of." "cents" is a form of "centum" which means hundred, and "c" was an abbreviation of "centum." Today we usually write the expressions as 30%, 20%, and 6%.

In order to determine the gain or loss on the goods he sold, a merchant would compare the value of goods he had as a hundred cents' worth of each kind of goods. Thus, to use our dollar to illustrate, if 10 dollars were given for 100 dollars' worth of one kind of goods, and 10 dollars were given for 100 dollars' worth of another kind, the merchant would then thought and write as "*10 per centum*" and "*10 per centum*." Thus, the two rates of gain would be reckoned. The English still write and speak of per cents in the same way. The expression, "25 per cent," meaning " $\frac{25}{100}$ parts out of 100," is common. In America, we do not write "*25 per cent*," but "5 percent"; and, instead of thinking of dollars out of 100, we think "5 hundredths." In comparison, we write 6% as 60, because we think of percents as hundredths.

It is often helpful, when the pupil has to find the percent of an amount, 6% of 300, let us say, to remind him of the older meaning, which is "6 out of a hundred." Thus, in determining 6% of 300, he may be encouraged to think: "If there must be 6 out of every hundred, and if there are 3 hundred, then there must be taken out 3 times 6, or 18." Of course, it is to be kept in mind that our present meaning of percents is hundredths; and that, to find 6% of 300, one may proceed to multiply 300 by .06.

2. The Percent Sign

The earlier form was "percento." Often, this form was abbreviated to "per c" and "pc." About three centuries ago, the abbreviation was frequently written as "per^o" or "p^o." From the latter abbreviation, we get our present form of the sign for percent, %, which is easier to write than the older forms. So long as one recognizes the sign, %, as an abbreviation for the expression that means "out of a hundred," or "hundredth," he encounters no danger of confusion.

Rates on bonds are often quoted as so much "per M," that is, "per mill," meaning "out of a thousand," or "thousandth." In Germany, the sign o/oo is used for "per mill," the sign being patterned after the sign for percent, instead of being derived from the form of "mill." Baseball averages are printed in "percents." In such case, the term 'percent' is not properly used, because what is really meant is "per mill." Thus, if a player has been at bat 100 times and has made 30 hits, he really has batted 30 percent. This, however, is printed as .300, which is the same as .30 or 30%. But, because the term 'percent' is familiar, and the term 'per mill' is not, the batting average of .300 is usually read as '300 percent' instead of '300 per mill.' So, when we speak of a player's batting average as his 'percentage,' what we really mean is his 'permillage.' If a player has been at bat 4 times and has made 4 hits, we say that he has "batted

a thousand." We should mean a thousand per cent. and a thousand percent.

XI. THE THREE KINDS OF FRACTIONS

Once the pupil has learned to deal with *common fractions* (that is, the fractions in common use), he has little yet to learn in his study of *denominations and per centage*. In common fractions, he knows to deal with fractions of several different sorts, as denominated, he has to give attention to but three or four — namely, tenths, hundredths, thousandths, and occasionally ten thousandths. In per centage, he needs to give attention to but one, namely, hundredths. In denominations, all that has to be learned is the use of fractions, presumably familiar, in a new form of representation — that is, now for the representing of *denominations*, and whole numbers — which also is presumably familiar. In per centage, all that has to be learned is the use of the *fraction* presumably familiar, in a new form of representation, which is merely the abbreviation of an expression that is well understood.

As the idea of the *fraction* develops, the pupil finds use for it in so-called 'practical applications' that is, both in comparing numbers and in making the comparisons between them. Whatever the manner of representing the *fraction*, as a common fraction, as a decimal, or as a per cent, the use of the idea in comparisons is exactly the same. The pupil's independence and confidence in such case depends upon his ability to distinguish the particular form of *fraction* of representation that is demanded by the practical situation.

The use of the idea of the *fraction* in making such comparisons is to be recognized as that one mentioned in "the three cases in per centage." In order to avoid the suggestion that such use is confined to per centage and to emphasize the suggestion that the use is common to all three forms of the expression of *fractions*, it will be referred to throughout this book as "the three kinds of *fractions*."

in fractions, in decimals, and in percentage. The three kinds follow:

First kind: Finding the part, or percent, of a number

Second kind: Finding what part, or percent, one number is of another number

Third kind: Finding a number when a part, or a percent, of it is known

Learning the methods of solution is never difficult. The methods may be stated briefly, as follows: First kind, *multiply by the part, or percent*; second kind, *divide*; third kind, *divide by the part, or percent*, it being understood that computations involving percents require the changing of the percent form to the decimal form. Learning how to make distinctions between the three kinds of problems is the difficult task.

The pupil may be trained to make the necessary distinctions. He may learn how to notice distinctions if he is given training and practice in looking for them. The second kind of problem is easily distinguished from the other two, but the pupil is often at a loss as to which number to put into the dividend. Once he knows that the "number being asked about" goes into the dividend and has had practice in looking for such number, he may be expected, when he has distinguished the second kind of problem, to ask the question of himself, "What number is the problem asking about?" Looking for such number may not be the equivalent of finding it; looking, however, is a necessary prerequisite. Moreover, whenever the activity of looking is unsuccessful, it at least makes understandable the assistance the teacher may need to give.

The first and third kinds of problems are easily distinguished from the second kind, but they are not readily distinguishable from each other. Once the pupil has decided that the problem in question is either the first or the third kind, he may proceed with independence, if he has been

tought the distinction between the two. It was never to think of the solution of the first kind of problem as: "If a part, or a percent, is given of a number that is given, multiply by the part, or percent"; and of the solution of the third kind of problem thus: "When a part, or a percent, is given of a number that is not given, find how to be found, divide by the part, or percent." With such sentences to assist the problem solver, the third kind may be attacked with an inquiry somewhat as follows: "To the part, or the percent, that is given in the problem of a number that is not given, what is not given?" The pupil may now be expected always to note the correct answer to the inquiry: "What is not given?" such an inquiry guides the pupil's steps.

CHAPTER VII

THE DEVELOPMENT OF ARITHMETIC

ARGUMENT

1. The general adoption of the Hindu-Arabic numerals stimulated interest in the art of computation.

2. Interest in computation resulted, not only in unusual combinations of numbers, but also in newer and better methods of computation.

3. Interest in problems developed originally out of real and practical situations. As such situations were attacked and were provided solutions, interest shifted from practical considerations to the puzzle feature of the methods of solution.

4. For the sake of the puzzle element, problems came to be sought in fanciful and impractical situations.

5. Such ascendancy of the puzzle interest confirms the point that a problem can provide a real interest even when its source bears no relation to reality.

6. The early interests in computations and in problems persist in present-day arithmetic, which divides into two parts; namely, computation and problem-solving. Ancient interests dominate the arithmetic of the modern school.

7. However, when ancient interests become too dominant, they produce critical reactions that are characterized by more or less sporadic enthusiasm. Chief among such reactions are:

- (a) The emphasis upon content and thinking, little as it was, in Warren Colburn's arithmetic of a century ago;
- (b) The elimination of outworn and useless topics of two decades ago in the interest of economy of time;
- (c) The selection of topics according to the criteria of social and business usage;
- (d) The tendency to motivate arithmetic through appeal to the immediate interests of pupils.

8. The method of elimination, which consists of subtracting one equation from another, is the simplest of the algebraic methods. It is used to solve a system of linear equations. The arithmetic of the method is based on the properties of number facts.

1. The following statements of present and former employees are pertinent:

- (a) The method of statistical analysis is not an aggregation of facts that have been previously understood in a logical manner.
- (b) Such analysis is that of an object rather than a subject, arithmetic, not that of a subject rather than an object, arithmetic; mathematics.
- (c) The acceptance of statistical analysis does not guarantee an understanding of those who use it.
- (d) Understanding of one, if present, is not a necessary condition for the use of statistics as a science.
- (e) The criterion of present and future progress is not the use of statistical analysis.
- (f) The interval between the use of statistics and the use of statistics is not the same, and the interval between the use of statistics and the use of statistics is not the same.

1. Form of Motion

[illegible]

II. COMPUTATION

1. Interest in the Process

The union of numeration and computation in the Hindu-Arabic system of notation, which made both processes an integral part of the written language of number, detached from mechanical devices yet retaining the virtues of a mechanical device, paved the way for the development of new interests in computation. Computations, such as multiplication and division, that theretofore had been carried through as additions and subtractions, respectively, now were made possible with the new place-system of numeration. With the new ability to carry through computations once looked upon as extraordinarily difficult, if not impossible, went the growth of new interests in computation.

Moreover, the new numerals provided a new interest in computation related to what may be called the speculative side of mathematics. From early times men had observed the strange properties that numbers possessed. They had at first associated these properties with magic; later, they undertook serious investigation of the principles underlying these properties. They observed the peculiar relations that existed between certain numbers; they studied the properties of prime numbers; and they on occasion organized in groups to determine the location of the last number. Though such speculations were the sport of philosophers, they were not entirely unknown to the popular mind. How greatly the popular mind has been impressed by speculations about number is indicated in the belief that still persists in the 'charm' of three, the 'magic' of seven, and the 'unlucky' thirteen. Because the new numerals were used to designate both units, tens, and powers of tens, a whole new series of unusual relations between numbers was revealed. Let us consider, first, how such speculations developed interest in computation, and, secondly, how interest in computation grew with the developing ability to compute.

2. The Magic of Numbers

The 'magic square,' shown herewith, distinguished the Great nine numbers in the squares so that they total 35 in every direction they are added. The square dates a thousand years before the Christian era. The square was copied by fortune tellers and soothsayers and used as a charm throughout the Orient. In the Middle Ages it was used in many parts of Europe to drive away demons and to bring good luck. It is found in many ancient & modern books, and it is said that every fortune teller and the Black makes use of it in his trade.

4	9	2
3	5	7
8	1	6

The properties of such magic squares have been the study of early mathematicians. Formulas for the building of magic squares of any given size have been evolved by the mathematicians of Japan,* and they have given their attention also to the building of "magic circles."

The 'strange' properties of numbers led to the construction of series of calculations like the following:

(a)

$$3 \times 37 = 111$$

$$6 \times 37 = 222$$

$$9 \times 37 = 333$$

$$27 \times 37 = 999$$

(b)

$$7 \times 11,875 = 831,250$$

$$14 \times 11,875 = 166,250$$

$$21 \times 11,875 = 249,375$$

$$63 \times 11,875 = 748,125$$

(c)

$$1 \times 8 + 1 = 9$$

$$12 \times 8 + 2 = 98$$

$$132 \times 8 + 3 = 987$$

$$1324 \times 8 + 4 = 9876$$

$$132456789 \times 8 + 9 = 987654321$$

(d)

$$1 + 9 = 2 = 18$$

$$22 + 9 = 3 = 116$$

$$1234 + 9 = 4 = 5216$$

$$12345 + 9 = 5 = 62316$$

$$123456789 + 9 + 9 + 9 = 951,623,416$$

* Dr. H. S. Gentry, *A History of Japanese Mathematics*, Chicago, Chicago, 1916, p. 172.

(r)

$$9 \times 9 + 7 = 88$$

$$98 \times 9 + 6 = 888$$

$$9876543 \times 9 + 1 = 88,888,888$$

$$98765432 \times 9 + 0 = 888,888,888$$

(s)

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

$$4 + 5 = 9$$

The foregoing calculations and others of like nature were early developments that retain an interest for the modern pupil in arithmetic. They are the forerunners of such modern curiosities as the following:

(g)

$$105263157894736842$$

$$\times 2$$

$$210526315789473684$$

Note in (g) that the same numerals appear in the product as in the multiplicand, but with the 2 moved from last to first place.

(h)

Set down the numerals 142,857 and multiply in turn by 1, 2, 3, 4, 5, 6, and 7. Note the same numerals appearing in the products with the exception of the last:

When multiplied by 1, the product is 142,857

When multiplied by 2, the product is 285,714

When multiplied by 3, the product is 428,571

When multiplied by 4, the product is 571,428

When multiplied by 5, the product is 714,285

When multiplied by 6, the product is 857,142

When multiplied by 7, the product is 999,999

The zero of the new system of numerals was itself a curiosity, and it influenced the development of computations that would produce curious arrangements of numerals and

4. Typical Methods of Multiplying and Dividing

Let us consider some of the methods and procedures of multiplication and division that were adopted one after the other. Each method had its day, only to be succeeded by another, until our present methods were adopted. Each as it came along contributed its share of interest to the art of calculation.

As previous discussions have indicated, multiplication at first was performed as a series of additions or by means of duplation. Later, a method of decomposition was used for those multipliers that could be factored. For example, to multiply by 42, one multiplied first by 7 and then by 6. Various forms for keeping the partial products in position were used when the multiplier was not factored. The following are from the Treviso arithmetic of 1478:²

$$\begin{array}{r}
 934 \\
 \times 314 \\
 \hline
 3736 \\
 934 \\
 2802 \\
 \hline
 293276
 \end{array}$$

	9	3	4	
3	7	3	6	4
	9	3	4	1
2	8	0	2	3
29	3	2	7	6

	9	3	4	
2	2	0	1	3
9	11	11	11	1
3	3	1	1	4
	2	7	6	

	9	3	4	
3	11	11	11	4
0	0	0	4	1
2	7	0	1	3
2	9	3	6	6

The multiplication shown in the four forms is $934 \times 314 = 293,276$.

² D. E. Smith. *History of Mathematics*, II (Ginn and Company, Boston, 1925), pp. 114-116.

Keeping in mind the easier multiplications, the following method of determining the harder ones was used:



To multiply 7 by 3, the numbers with their complements to 10 were used. Then the products of the complements, 3×2 , gave 6; and the difference of one factor and the complement of the other, $3 - 3$, or $7 - 2$, gave 5, thus the answer 56 was determined.

Division was at first performed either by successive subtractions or by 'mediation'; that is, successive halving. Later, the factors of the divisor were used as mediators. To divide by 24, one divided first by 4, then by 3, or first by 3, then by 8. The following will illustrate a few of the procedures used when the divisors were not factored.

To divide 900 by 8:

$$\begin{array}{r}
 10 = 2 \times 5, 20 = 4 \times 5, \dots \\
 \begin{array}{r}
 900 - 160 \\
 \hline
 160 \\
 160 - 80 \\
 \hline
 80 \\
 80 - 40 \\
 \hline
 40 \\
 40 - 20 \\
 \hline
 20 \\
 20 - 10 \\
 \hline
 10 \\
 10 - 5 \\
 \hline
 5 \\
 5 + 5 = 10 \\
 10 - 2 \\
 \hline
 8 + 8 = 16
 \end{array}
 \end{array}$$

To divide 1728 by 12:

$$\begin{array}{r}
 24 \\
 1728 \overline{) 144} \\
 1728 \\
 \hline
 11
 \end{array}$$

To divide 1728 by 144:

$$\begin{array}{r}
 12 \\
 1728 \overline{) 144} \\
 1728 \\
 \hline
 14
 \end{array}$$

The following method is the forerunner of our present method of long division:

$$\begin{array}{r}
 25 \overline{) 625} \quad 25 \\
 \underline{4} \\
 22 \\
 \underline{10} \\
 125 \\
 \underline{100} \\
 25 \\
 \underline{25} \\
 0
 \end{array}$$

The method appeared in the fourteenth century and is not unlike the method in use today.*

III. PROBLEMS

1. Interest in Problems

The arithmetic problems of early times, like those of the present day, originated in the real situations of life; and, like those of the present day, they took on modifications in terms of the special interests they helped to create and in terms of the special interests of the people who made use of them. The length of time it would take for water flowing through a pipe to fill a cistern, and the length of time it would take an army to march a certain distance were, for example, real situations in the lives of early peoples. They were problems that on occasion possessed an intensely practical importance. It was necessary for the people of the time to face the issues raised and to meet them to the best of their knowledge and ability. As the issues were faced from time to time and as solutions were reached by one manner or another, interest shifted from the practical issues involved to the logical issues. The use of a given method of solution served to attract attention away from the practical issue and to set the method up as a center of interest. The method involved gradually grew in interest, and, in later

* See Smith, *op. cit.*, pp. 128-144.

years, when the original issues or situations had been superseded by newer ones, it influenced, for the sake of its own preservation, the substitution of fanciful issues and situations for the original genuine ones. It happened moreover that the people who had to face the original issues of the practical situations met them through the use of various computational devices. Now, when such devices preponderated in the solution of the problems, it was bound to happen that the original problems and methods of solution would absorb, or be influenced by, the interest in computation. Thus, a problem *in the real world* was transformed into a fanciful one, and a situation that once consisted of simple number relations would develop into one with involved number relations.

Moreover, problems have always possessed a puzzle interest, which has manifested itself in many ways. One likes a problem that mystifies a little or amuses. The puzzle provides a game to play; it suggests a satisfaction of itself. It is calculated to leave the solution, when once reached, still a bit in doubt; its main business is excitement. It easily overshadows all interests that may be genuine and real, and when computation is difficult and uncertain, it readily takes advantage of computational difficulties to add to its power to mystify.

One who becomes absorbed either in the logic of his method of solution or in the puzzle phases of his problems easily ceases to be concerned with any practical value that his problems may have. He quickly loses touch with the real world, and, if he is a teacher of youth, he may ignore their immaturities and the closeness of their connection with reality. To him, the problem is the thing, he does the problem is the center around which all else must revolve. To him, the problem is important in its own right only. The fact that the logic of his method is a matter of consequence often serves as a hindrance instead of a help in bringing in mind the significance of its practical applications.

2. Illustrations of Problems

Following are some illustrations of the types of fanciful, computational, and recreational problems that absorbed the attention of early students of arithmetic:

Say quickly, friend, in what portion of a day will four fountains, being let loose together, fill a cistern, which, if severally opened, they would fill in one day, half a day, the third, and the sixth part, respectively? ¹

From real origins, the problems soon passed into the pseudo-real class. Alcin cites the case of a snail which took 246 years, 210 days, to get to a banquet; Mahavira has a lame man walk for three and one-fifth years at a time, and he pictures a snail crawling up a mountain.²

A man whose end was approaching, said to his eldest son, "Divide my goods among you thus: You are to have one bezant and a seventh of what is left." Then to his next son, he said, "Take two bezants and a seventh of what remains." To the third son, he said, "Then you are to take three bezants and a seventh of what is left." Thus he gave each son one bezant more than the previous son and a seventh of what remained and the last son had all that was left. Moreover, after this division, it developed that they had shared the father's property equally, although they had followed out his conditions. The question is, how many sons were there and how large was the estate? ³

A lion is in a well whose depth is 50 feet. Every day, he climbs up $\frac{1}{2}$ of a foot and slips back $\frac{1}{3}$ of a foot. In how many days will he get out of the well? ⁴

A mouse is at the top of a poplar tree that is 60 ft. high, and a cat is on the ground at its foot. The mouse descends $\frac{1}{2}$ of a foot each day and at night it turns back $\frac{1}{3}$ of a foot. The cat climbs 1 foot a day and goes back $\frac{1}{2}$ of a foot each night. The tree grows $\frac{1}{4}$ of a foot between the cat and the mouse

¹ L. C. Karpinski. *The History of Arithmetic* (Rand, McNally and Company, Chicago, 1925), p. 46.

² V. Sanford. *A Short History of Mathematics* (Houghton Mifflin Company, Boston, 1930), p. 218.

³ *Ibid.*, p. 219.

⁴ *Ibid.*, p. 207.

each day and it strikes $\frac{1}{2}$ of a foot every night. In how many days will the cat reach the mouse and how many will the tree grow in the meantime, and how far have the cat and climb?

Another famous problem was the one concerning the number of grains of wheat that man, theoretically speaking, he placed upon a chessboard, one grain being put on the first square, two on the second, four on the third, and so on.

AD: 11, 12, 13

Expenditure for



only the number of grains but also the number of crops necessary to carry the total amount, the value of the crop, and the impossibility that all the countries of the world should produce such an amount of wheat."

The following problems are to be found on the organization of the century just passed:

Three men lived together, one of them drank 12 mugs of beer a barrel of cider alone in 4 weeks, the second man drank it alone in 6 weeks, and the third in 7 weeks. How long would it last the three together?

A cistern has 3 cocks to fill it, and one to empty it. One cock will fill it alone in 3 hours, the second in 4 hours, and the third in 6 hours. The other will empty it in 3 hours. If all the cocks were allowed to run together, how long would it take to be filled? "

An arithmetic of our own day inquires of the sixth-grade pupil about the number of tons of stone that Egyptian masons have hauled in 900 years, working 300 days each year to a city 20 miles away, on an cart that would carry 1 ton of stone, provided his oxen travelled 2 miles an hour and he spent 10 hours a day travelling with them.

* Ibid., pp. 207-208.

See also Smith, *op. cit.*, pp. 644-645.

⁹ *Ibid.*, pp. 243-250.

¹⁰ Warren Colburn; Hilliard, Gray and [redacted]
¹¹ Ibid., p. 298.

11 *Ibid.*, p. 239.

The following problem, which never fails to evoke interest, is an illustration of modern problems of the "recreational" type:

Two clerks start in an office at the same time, one at a salary of \$1000 for the first year and a raise of \$200 each year thereafter, and the other with a salary of \$1000 a year but with a raise of \$50 every half year. Which has the larger income? ¹²

IV. LOGICAL VERSUS PRACTICAL MOTIVES

There is a theory current in our schools to the effect that problems have to be 'real' problems, drawn from and applying to the practical side of life, in order to arouse the interests of pupils. The logical motive for attacking problems is frequently given no more than passing attention. Let a problem be drawn from the everyday experiences of people -- preferably, of the pupils themselves -- buying at the store, making change, determining the standing of a ball club, figuring the cost of a radio, working in the school garden, etc., and the worth of the problem is established beyond question. But let a problem be drawn from a fanciful situation, from one artificially produced for the sake of the lesson, from one which varies the slightest from reality, in order that an idea, or a number relation, or a principle of procedure may be illustrated and emphasized, and the problem is subject to severe condemnation. Is this true? Must the problem always be a real one? Must the situation that produces the problem always be actual? Which is the factor of central importance -- the problem, or the method of thinking illustrated? Moreover, what kind of problems are effective in evoking and stimulating the interests of pupils? Which is the more successful in commanding the attention, the practical or the logical?

To the questions just raised we must withhold our answers, reserving them for a later discussion. Or, what is more to

¹² D. E. Smith and W. D. Reeve. *The Teaching of Junior High School Mathematics* (Ginn and Company, Boston, 1927), p. 390.

the point, let us suspend judgment about problems in arithmetic until we have made further progress in our discussion about the nature and development of the subject. We will do well for the present, however, if we will reflect that it was the logical interest in methods of procedure, rather than the practical interest in real situations, that determined the kinds of problems engaging the attention of mathematicians for more than twenty centuries. In order to proceed and to pass along the kinds of fanciful, unreal, organizing, and computational problems that have just been sketched, the arithmeticians of the past had to possess an interest that transcends the practical.

All this is intended, not as a plea for the abandonment of 'real' problems in our present-day classes in arithmetic, but as a plea for reflection upon the organization of our procedures.

V. RESULTS OF ANCIENT INTERESTS

The two kinds of early interests just described — interest in computation and interest in problems, have continued to the present day. They expanded as long as to become what may be termed a racial habit of mind, or attitude, or point of view toward the subject of arithmetic. However isolated the two interests were in the thinking of early mathematicians, they finally developed as two separate and distinct interests. Computation, calculation, reckoning, was one thing, being able to understand and to solve the problems was another thing. Each interest had a peculiar terminology, methods required, or seemed to require, a different attack, and each developed, as we have seen, from a different source.

We find in our schools today the results of these two kinds of early interests, which have been inherited from the past and which are being constantly passed along by teachers and by writers of textbooks. In our schools, computation is still one thing and problem-solving another — as they have

the pupil to perform the operations of arithmetic, and next we teach him how to solve problems; the pupil first must learn the 'number facts' and combinations, and then he is expected to 'apply' them; each chapter of the textbook first provides exercises for drill, and next some problems to be solved; ability to compute is the first objective, and ability to solve problems is the final objective. For the two kinds of activities to be performed, or of things to be learned, or of goals to be attained, we have succeeded in developing different techniques of instruction. We have formulated and seek to carry into practice two distinct methods of procedure — the one adapted to computation, the other to the solution of problems. Computation and problem-solving persist in our schools as separate activities and as distinct interests.

This modern view of arithmetic, which separates the subject into two distinct parts, may be the correct one. Whether correct or not, however, we ought to recognize that the view was determined by the accidents of early developments and has been handed down to us ready-made from the past. If we can recognize the view as one that was determined by early enthusiasms, we may bring ourselves to the point where we will be inclined to subject it to critical examination and analysis.

VI. REVOLT AGAINST ANCIENT INTERESTS

1. Revolt Against Extremes

While the view that arithmetic embodies a double set of interests still persists, there have been sharp and violent reactions against the earlier tendencies to carry such interests to extremes. The first reaction began a century ago with the publication of Warren Colburn's arithmetics. The following enthusiastic account of Colburn's work, published a half-century later, will convey an idea of his influence and of the reaction he fostered:

Fifty years ago, arithmetic was taught as a mere collection of rules to be committed to memory and applied mechanically to the solution of problems. No reason for an operation was given; none were required; and it was the privilege of only the favored few even to realize that there is any thought in the process. Amidst this darkness a star came in the East; that star was the mental arithmetic of William Colburn. It caught the eyes of the scholars and led them to the new life of the mind. The old arithmetic was like a dead body, and beauty to all the various parts."

The following quoted paragraphs trace the development of arithmetic from Colburn's day down to the beginning of the present century:

For three-quarters of a century after William Colburn published the first distinctively American arithmetic in 1822 there was a steady expansion of the content of the subject. Denominative numbers came to be treated as a separate practical part of the subject, and the treatment of their uses was greatly enlarged. Arithmetic was given an increased time in the school program. Arithmetic may now properly be described as a favorite subject in the American school which it has been for many years. It is a subject in which the attainments of pupils can be accurately determined. Pupils and teachers alike accepted it as an important part of the school work and were satisfied to have it occupy an appreciable part of the school day.

¹¹ Edward Brooks, *The Philosophy of Arithmetic*, Boston: Houghton Mifflin Company, Philadelphia, 1876, p. 11. C. H. Judd, *Summary of Arithmetic*, (Department of Education), p. 181.

During the later decades of the nineteenth century there came a general change in the conception of education. Society began to demand a broader type of schooling. The result was a general expansion of the curriculum. This expansion raised numerous questions as to the possibility of making room for new subjects by eliminating waste in the traditional courses of study. Education began to question every topic included in the curriculum.¹¹

2. Elimination for Economy

The axe of elimination¹² fell upon the subject of arithmetic as well as upon other subjects in the curriculum. "Minimum Essentials," "Economy of Time," and the like were the catch phrases of educational discussions and writings. Such topics as the following were shorn from the curriculum in arithmetic in many of the schools of the country:

1. Long method of G. C. D.
2. Most of L. C. M.
3. Long, confusing problems in common fractions.
4. Long method of division of fractions. (Always invert and multiply.)
5. Complex and compound fractions.
6. Apothecaries' weight, troy weight, the furlong in long measure, the rood in square measure, the dram and quarter in avoirdupois weight, the surveyors' table, the table of folding paper, tables of foreign money, all reduction of more than two steps.
7. Most of longitude and time.
8. Cases in percentage. (Make one case by using x and the equation.)
9. True discount.
10. Most of compound and annual interest.
11. Partial payments, except the simplest.
12. Profit and loss as a separate topic.

¹¹ Buswell and Judd. *Op cit.*, pp. 9-10.

¹² See *Fourteenth Yearbook of the National Society for the Study of Education, Part I.* (Public School Publishing Company, Bloomington, Illinois, 1915.)

13. Partnership.
14. Cube root.
15. The metric system.¹⁰

3. The Criterion of Social and Developmental Change

The criterion for the selection of the subject for the standard in the curriculum in arithmetic was social and developmental change. Wilson, who has been a leading advocate of these principles, defines it as follows:

This principle insists that subject matter chosen should be fundamentally useful from the standpoint of the needs of society. Interpreted in terms of child life, it means that the child shall see and understand concretely and the usefulness of what he is learning. And this means that the child is able to organize into his own thinking and acting the things upon which he is working in school. Formally expressed this principle that school topics and subjects shall be useful to society means, on the one hand, that they shall not remain adult usage, and, on the other hand, that they shall be adapted to the present thinking and understanding of the child.¹¹

Again he remarks:

... it is assumed that arithmetic in the grades is justified only on the basis of its utility in the common affairs of life. We learn the multiplication table, and to measure the width of to comprehend a beautiful system, but to figure our bills, our taxes, or the interest on a note. Arithmetic education is given in the grades beyond the measure required by social utility consumes time that could be used more profitably in other ways.¹²

¹⁰ G. M. Wilson and others. *Commonwealth Course of Study in Mathematics for the Elementary Grades* (Warwick and York, Mass., Houghton Mifflin Company, 1922), pp. 13-14.

See also W. A. Jenson, "The Arithmetic of the Elementary School Teacher, 14: June 1914.

¹¹ G. M. Wilson. *Elementary Mathematics* (Boston, Houghton Mifflin Company, 1914), pp. 1-2.

¹² G. M. Wilson. *Elementary Mathematics* (Boston, Houghton Mifflin Company, 1914), pp. 1-2.

The method by which one uses the criterion of social utility to determine the curriculum is simple, though tedious and time-consuming. One first discovers what phases of arithmetic are being used by different social groups and what phases are not being used. The next step is to include the 'useful' and eliminate the 'useless.' Thus, if one discovers that Case III of percentage, for example, is not being used by adults, he discards that topic from the curriculum.

4. Motivation by 'Immediate Interests'

The revolt against ancient interests in computation and problem-solving has in recent years expressed itself in another way. One must not only eliminate the 'socially useless'; he must also include the 'useful' in terms of the immediate interests of pupils. The child must be made aware of, and he must be led to understand, the usefulness of the process or procedure that the school has set for him to learn.

Instead of leading the child blindly through years of meaningless abstract problems, we have come to follow a child's interests and needs in selecting problems related to the home, the school, and the community. The method of presenting these conforms to the psychology of learning. The recognition of the instincts for play and activity and of the laws of interest, association, and habit formation are clearly shown in both subject matter and methods. Number concept, ideas, and process are built up through plays, games, dramatization, and motivated activities. At first, in the lower grades, problems growing out of the child's immediate environment build up and enlarge the application of number ideas and processes and aid as a background for the formal work. Later in the grammar grades community problems of a more complex nature give still wider scope to the mathematical training and lead the child out with a more efficient equipment into the world about him.¹⁹

¹⁹ Katherine L. McLaughlin. "Summary of current tendencies in elementary-school mathematics as shown by recent textbooks." *Elementary School Journal*, 18: March, 1918, pp. 543-551, esp. 545.

listed, labeled, numbered, and otherwise set forth, each as a separate and distinct entity. Arithmetic no longer could be considered an entity; henceforth it must be considered a composite of a multitude of details.

The analysis of arithmetic has led to the listing of 390 separate simple 'combinations,' 765 higher-decade additions, an equal number of higher decade subtractions, 40,095 two-digit divisions with single-digit quotients, to mention only a few. To be sure, not all the facts so listed are found to be 'socially useful.' Such may be eliminated, and the 'useful' facts retained in the curriculum. Since those 'facts' that are not 'useful' may be summarily dismissed from the curriculum, it follows that they have no logical connection with those that are retained for reasons of social utility. In other words, no relationship of any consequence exists between the 'facts,' whether 'useful' or 'useless.' Likewise, the 'useful facts' are of importance each in its own right — because of their 'usefulness' — not because of any relations that may exist among them. And so, because of the characteristics assigned to them, the 'number facts' must be taught to children each as a separate and distinct item of experience. One may find in the work of the school some strange and curious anomalies, but none more strange and curious than the union of the doctrines of social utility and of appeal to children's interests with the analysis of arithmetic into a multitude of unrelated details.

VIII. CRITICISM OF PRESENT TENDENCIES:

FACTS ABOUT NUMBER FACTS

With respect to the theory that arithmetic is a composite of many discrete isolated facts, it is important for one to consider (1) that an analysis of arithmetic into its parts may be entirely unrelated to the manner by which children learn the subject, and (2) that any such analysis is the analysis of the adult who has already learned the subject and is the product of adult interests rather than of children's inter-

ests. The adult may undoubtedly discover the various 'facts'; he may think of them in their relations, indeed, the last step for him is to put arithmetic into its proper perspective and to see it as a unity. But he recognizes that it is not possible to discover arithmetic as a unity without first seeing the facts one by one as just as separate facts as the ordinary man will be able to understand the running automobile when shown a half hour or so in the garage there. The mechanic who has repaired these 'parts' recognizes them as distinct parts, in the same way that he recognizes them in relation to each other and to the machine as a whole. He recognizes a separate part but he is able to do so only because the separate parts of the machine as a whole require a separate and distinct use. The separate uses are revealed, and by the parts themselves as well by the larger idea of unity that brings them together. The adult who is so enthusiastic about analyzing things, if, when analyzing arithmetic, he recognizes the parts, would also make an analysis of the 'facts' themselves, if he is going to engage in analysis. He might, for example, question his procedure of analysis, if he recognizes that his analysis when he has reached the 'conclusion' is only a view a 'fact' from more than a single angle.

At least three facts about a 'number' fact are worthy of consideration. There is, first of all, the existence of the 'fact' "Three and two are five" as a fact that even a savage could count and that even a child learns it or not. The child may be taught its existence, and may accept its existence as the only way of other things, through his confidence in the teacher and authority of the teacher. With most children, however, the second fact of major importance, namely, the understanding of the number fact, is learned from the teacher. The number is learned as a mere conventionality. Through practice even the youngest

may learn the statement. The third fact of major importance is the *consciousness* of the number fact. At this point intelligence plays a part, and at this point a contribution to the child's intelligence may be made. Consciousness does not of necessity follow the acceptance of existence and the learning of the statement, since these two may take place without any relation to 'consciousness' — in the sense of 'intelligent comprehension.' Consciousness, however, includes the other two. One may with propriety raise the question whether a child can be made 'conscious' of a 'number fact' when it is so presented as to exist in his experience merely as an isolated unrelated item. The experience of teachers is very revealing on this point. They frequently find that, after a pupil has accepted all the 'useful number facts' and learned how to state them with ease and with accuracy, he does not know what to do with them. They frequently find, too, that he is more hopelessly confused after he has learned many of the so-called 'facts' than he was when he knew only a few of them.

With respect to the theory that the work in arithmetic must be connected with games and plays and other children's interests in order that the learning may be motivated by an understanding of usefulness, it may be well to remember that in the development of arithmetic the invention of devices always preceded the necessity for their use. One may reflect, if he wishes, that modern industrial society would break down if the Arabic numerals and the methods of thinking that they facilitate were suddenly withdrawn. It is, however, more to the point to reflect that modern industrial society did not come into existence until the Arabic numerals had been invented and accepted as a means of number-thinking. Moreover, one may raise the question how a child can be made 'conscious' of the 'usefulness' of a process in arithmetic before he learns it. Suppose it does relate to his interests, or what the adult views as his interests; how is it possible for the child to see and to understand

such relation before he has learned the grammar? But this process means, what its importance as well. Some of the ideas used are learned only as the process is learned. In other words, they sometimes are not learned at all as the grammar is learned. How, then, can the usefulness of the grammar be a motive for learning it? One of the most striking facts in connection with falling in with the idea of "grammar as the first and untrivial part of the study of language" is that it has been and is being used in the schools of the United States as a means of learning to read and write.

With respect to the theory that social and business usage shall determine the content of the curriculum, it will be well for one constantly to remind himself that this theory has worked hand-in-glove with the theory that curriculum is a composite of isolated unrelated items. Where usefulness is the criterion, any idea of a relation between various social processes must be abandoned. When social and business usage be included or excluded at will, any relationship existing between them must be decidedly unimportant. It is true that a social process may not be "useful" today, but only today. It may be ten years from today. What curriculum should be based on? To teach the topic or process after it ceases to be useful to people who are to use it is to waste time. It is the duty of the teacher to teach what is useful.

The exclusion from the curriculum of the study of Case III in present-day teaching is a result of the fallacy of the social usage theory. Case III was apparently not socially useful. It was abandoned from the curriculum in many schools, while Case I, which was found to be useful, was retained. The prevailing idea of study of Case III might enlarge the popular view of percentage in general and thus make Case I a more intelligent user of Case I was not considered. Moreover, as the study of all the agitation about the "usefulness" of Case III American wholesalers and retailers very unhesitatingly began to use it, while they took up the practice of using Case I.

profit as a percent of the selling price rather than as a percent of the cost — thus, of course, bringing rapidly into use the procedure of finding a number when a percent of it is known.

The short-sightedness of the theory of social and business usage becomes apparent when one raises the questions: What makes a useful fact useful? Is the usefulness of a fact carried by the fact itself? Or is a fact useful only as the intelligent person finds use for it? One may discover in our schools many pupils who learn useful facts, but are unable to use them, and one may discover many pupils who are constantly finding use for the things they learn. Perhaps by reducing the curriculum to the useful and the practical we may neglect to impart many things that, though useless in social and business practices, may be of exceeding usefulness in the training of pupils. In physical education classes we teach the youth many plays and games. We do not expect him to find these quite so useful as he grows older; indeed, we expect him to abandon the games of childhood. The plays and games are taught, not because they are useful in later life, but because they furnish a training that will be useful. The arithmetical method of extracting the square root is certainly not a useful device in adult life. Most people have no occasion to use the device, and those who do have such occasion find it easier to resort to the table of logarithms or to the slide rule. Why, then, shall the school teach the extraction of the square root? It may be that the understanding of the method will give the pupil a clearer insight into the spatial relations of square measure.

IX. COMPUTATION AND PROBLEM-SOLVING, THE PERSISTENT CONCERNS

In outward appearance, the modern textbook differs markedly from textbooks of earlier days. In externals, modern methods of teaching differ markedly from those of our fathers' time. In content, the modern curriculum differs markedly from the curriculum of the past. Internally, how-

ever, the influence of the post-war period in the school. Computation and problem-solving are the main concerns of earlier days, computation and problem-solving are the main concerns of the present. The changes in the earlier concern in computation and problem-solving have led to sharp reactions on both sides of the computation and the kinds of problem-solving. The changes have been radical changes; but the changes have been changes in procedure, and changes in the effort. The interests of earlier periods have been different. Instead of moving back to the main road, the greater part absorbed our attention in detecting what we thought to be better and more powerful methods along the side road. The following statement, which is perhaps somewhat extreme, is not without reasonable excuse:

For finally the student can never get beyond the system. Having no insight whatever into the system, unable to reason and generalize from it, he has no assurance that his results are correct, and he never correct them, in the proper sense. All he can do is to substitute one method for another. And so we reach the paradox that the student who has been regarded as the best of all pupils in the school of the reason has come to be the student of the system where the exercise of the reason is most difficult.

"McQuilkin DeGrange. 'Mathematics and Psychology.' (Educational Administration and Supervision, December, 1931), pp. 561-573, esp. 566-573.

CHAPTER VIII

METHODS OF THINKING

ARGUMENT

1. The sketch of the development of the number system in preceding chapters does not suggest that the child in school must develop his ideas of number according to the methods the race had to use. It merely describes the number system the child must learn.

2. The child must learn to think the arrangement of objects in the operations of addition, subtraction, multiplication, and division according to the pattern fixed by the race.

3. The issues and situations incident to number combination that confronted earlier generations now no longer exist.

4. Such issues and situations ought not to be artificially reproduced by the school for the sake of raising problems to be solved. To do so is to follow a procedure in which there are four fallacies:

- (a) the fallacy of time;
- (b) the fallacy of inferring the child's point of view from the adult's;
- (c) the fallacy of paralleling the learning situations of the child with those of the race;
- (d) the fallacy of inconsistency.

5. The perfection of the number system has removed the element of doubt from situations having a number phase

6. Since a problem is a question involving doubt, when the doubt is removed from a question there is no problem.

7. To keep arithmetic as a set of problem activities is to withhold information and training in order to provide doubts. The school can hardly justify itself when it follows such antiquated methods of procedure.

8. The purpose of arithmetic in the school is to teach the

number system so that it will be understood and so bring the system in doubt for the sake of problem-solving.

9. An understanding of the number system, as and so be derived from explanations that take their origin from the child's experience with number. Such explanations do not encourage self-confidence and self-reliance.

10. A true understanding is to be developed only in terms of the development of general ideas on the part of the child. The organization of arithmetic, if performed, produces a constant portrayal of such ideas.

11. The procedure of learning is neither one of attempting to explain nor one of trying to master the number system. Number situations that are left obscure, but which are of developing a method of studying and attacking such situations.

12. The problems of arithmetic should not be presented as such, but as illustrations of ideas and methods that are really important.

ORIGINALS IN THE HANDS OF THE AUTHOR

I. RELATION OF THE ORIGINS TO LEARNING OF ARITHMETIC

Before we turn to a discussion of the stages of development in number which the school should know and through which the child should pass, it may be well to sound a note of caution. Our rapid sketch of the stages of the number system is intended to bring to its present stage of particular the number system in use today is intended to be a guide to the suggestion that the child must develop his number system according to the same methods that the teacher has used. Our sketch is intended merely to indicate some of the characteristics of the number system that we have just described for its use and that it expects the child to acquire in order to take his place as a member of society. In this sketch the development of the various methods of arithmetic has been made upon the number situations mentioned above, in order to discern, if possible, the methods that have been most effective; and to give the way for a consideration of the place arithmetic has made for itself in the teaching of school and of the purposes of learning in arithmetic.

It is easy to imagine a parallel between racial development and individual development, and to set up an argument that the conditions and situations that were instrumental in furthering racial progress should be duplicated in order to provide the motives and the settings for the progress of the individual. Indeed, as the preceding chapter has indicated and as our present discussion will show, it is extraordinarily difficult for the present to break away from the influences of the past and to make adaptations to present conditions. The attempt to duplicate the situations that confronted earlier societies in order to furnish the motives for learning in present society is, however, a distortion both of modern situations and of the conception of the true motives for learning. Though the child must learn the number system the race has evolved, the conditions that surround him are radically different. He has to learn in a short time what it took the race a long time to develop. He must be steered clear of the mistakes the race made. He must not be allowed the long periods of experimentation with inadequate methods of thinking that the race through accident had to experience. He is surrounded at birth by a perfected number system in daily use by the older generation, and throughout his formative years, if not his whole lifetime, he may profit through such experience; whereas the race enjoyed no such advantage because it was at all points confronted with the task of creating a number system and bringing it to perfection.

The only item in the parallel is the number system itself. Through blind trial extending over long centuries the race succeeded in creating and developing a number system; it is this same number system that the child must re-create and redevelop under the systematic guidance of the older generation. The race lacked guidance and consequently fell into many difficulties; the efforts of the child may be guided in the direction of the successful methods of attack that the race through trial and error finally came to adopt.

II. THE FUNDAMENTAL

The operations of addition, subtraction, multiplication, and division, which are learned after other operations by pupils in the early years of their school life, are referred to as illustrative (1) of the fact that the race have fixed the patterns of their thought, and must learn to think the arrangements of the world as they are, the fact that the world is what it is, and that the world should not now be artificially rearranged. The operations to the first type of illustration is as in the operations of addition, subtraction, multiplication, and division, the operations of addition, subtraction, multiplication, and division, add and to multiply do not mean the same thing as subtract and to divide do not mean the same thing as multiply, to regroup, and, when the number is greater than nine, to regroup, to regroup, to regroup, to regroup, of ten. Let us consider these four operations as they are.

First, addition does not mean to put together. Eight and seven are fifteen, yet a difference exists. This is of course, not in the number of objects, but in the arrangement. number remains throughout the operation. When adding seven and eight we simply add the two numbers together into a new arrangement. When we add a number of ten and five. The standard arrangement is not comprehended than the standard arrangement. The former may be judged in terms of the latter, and is of standard size.

Second, subtraction does not mean to take away. Ten minus eight equals two. Here we have a new arrangement. We rearrange the ten into a group of eight and two. This is at least as a matter of thought. When we subtract eight and finding two, we deal with the same objects, but we deal with the same operation, finding them first, and then subtracting.

of ten and five and lastly in the chance arrangement that is demanded by the situation at hand.

Third, multiplication does not mean to increase. Three fives are fifteen. We start with three groups of five and five and five. The standard arrangement is one ten and five. Six sevens are forty-two. The economy of the rearrangement is that in the one instance we have a certain number of objects in a chance arrangement, and in the other, the latter, instance the same objects are thought together in the standard arrangement. We simply clarify our thinking about them by standardizing the grouping into four tens and two. Economy is further illustrated by the necessity for comparison. Suppose, for example, we need to compare six sevens and five eights. In the original arrangements, comparison is very difficult and consequently subject to error. The standard arrangements, however, are easy to compare—four tens and two compared with four tens.

Fourth, division does not mean to decrease. Forty-two divided by six equals seven. In this instance, we merely translate the standard arrangement of a given number of objects into the arrangement demanded by the necessities of the moment. The question is, in forty-two, how many groups of six each? or in forty-two, how many do we have in each of six groups? Ten divided by one-half equals twenty. The process of division is not reversed. Ten is not increased to twenty. The question is, how many halves are there in ten? The answer is, twenty (halves).

There is nothing natural about the grouping of objects into tens. The writer remembers the little boy who described his wealth in the following terms: "I have a quarter, five nickels, three dimes, and seven cents." The boy was not aided by the suggestion, "How much do you have altogether?" The boy had already thought his money together, and had described the total. The suggestion of eighty-seven cents as a statement of the total was not helpful. The boy learned at a later date to group in tens, when the

school had taught him the system of grouping and that he had perfected and made a part of his own mathematics.

III. OPERATIONS WITH NUMBERS

With respect to the second type of mathematics, as previously allusion has been made, it may be suggested that there was a time when the methods of dealing with groups were not at all, as problems requiring study, the exercise of reason and critical judgment, and not as mere devices and tricks understood methods of dealing with the quantitative relations of life. The time was, as our preceding chapters have indicated, when the manner of doing things was the same as a group raised grain and charged expenses. The question after another was tried - giving attention to the results of the members of the group, making the expenditure of the group in correspondence with labor, etc. The time was when the group left the question still in doubt and the question of the cost of the group still undecided. The time was when the necessity of counting a large group rather than a small one was a device then another, and, after enlarging the group on the one hand, the enumerator was still an expert about the group, and the group remained an undivided and undivided whole. The time was when the fundamental operations of addition, subtraction, multiplication, and division were in the service of the expert, because the groups of numbers could be regrouped in the way considered to be most or less a mystery to the common man. A mathematician would have faced the necessity of dividing his stock into shares or a merchant his goods. The necessity was not at all perfectly clear, but the manner of doing things was not a question presented a problem. Mathematicians would be asked to solve it in various ways, but there was no common way in which the solution could be reached with accuracy. When a method of solution was used and a solution given, the solution was a doubtful one, and the original question of the group still remained an undivided whole.

Such problems as have been mentioned now no longer exist in civilized society. If one wishes to know the number in a group, he counts. If the necessity of arranging groups into larger ones, or of separating larger groups into smaller ones, presents itself, one knows exactly what to do and how to do it, whether the process demanded be addition, multiplication, subtraction, or division. Civilized society is in possession of methods of attack upon groups, methods of thinking their arrangement and rearrangement, that primitive society did not have. The methods of attack and the methods of thinking have worked to remove the element of doubt from the quantitative situations of life. With the doubt removed, the issues that confront one are no longer problems to be solved, but tasks to be performed.

It frequently happens, however, that the school undertakes to introduce the operations of addition, or of percentage, and the like by surrounding the child with the situations of life that seem to require the use of the operations next to be learned. The purpose is to provide a motive for learning. Let the child play the game of bean bags or be a clerk in the school store and in the course of time he will see the need of addition. The game or the clerkship becomes a problem that is unsolved until addition is learned. The game or the clerkship serves finally to direct the child's attention away from itself to the activity of addition, and this before addition is introduced. In other words, the attempt is made to withhold the introduction of addition until the child is brought, by some means or other, face to face with the problem of thinking things together in the orderly and systematic way that addition provides. Or, to state the matter in another way, the attempt is made to reproduce in the modern school the type of problem situation that in a primitive civilization slowly but finally led to the creation and use of addition.

There are at least four fallacies in such procedure.

One is the fallacy of time. The school life of the child is

too short to permit the duplication of the error previously made in the performance of such a similar operation.

A second is the fallacy of assuming that the child's point of view with respect to the situation of quantities and lengths may be inferred from the adult's point of view. The evidence is perfectly clear on the point that he is not concerned with quantities and the clerkship in the same way as the adult. The use of the operations of addition, the child's point of view has been introduced to children suddenly since such operations are the adult's superior point of view.

A third fallacy is the fallacy of generalizing the experiences of the child with the increasing sophistication of primitive people. The child is no doubt a primitive in the sense of the term, but he does not give us a picture of a primitive and, what is just as much in the case, the primitive does not reproduce the conditions of civilization. He is just about as modern, though perhaps not so sophisticated as serious, to deprive the child of merely a "primitive" way of dealing with groups until he assumes the modern, but what he would be to deprive him of modern knowledge until he arrives at the point where the modern way is better.

A fourth fallacy is the fallacy of generalizing. The situation is presented in order to consider the child's point of view in a problem requiring addition but we may find no evidence. We have having inferred that the child has assumed this position, just as the adult conceives that he should be in a position to learn to the learning of the process of addition that is a better solution. In the first place the child may be in a position to add in the second place, though he may be in a position to add as a means of making a generalization. In the third place, the adult may insist that the process be learned as a means of solution, for without a generalization, there can be no solution. The result is that, when the child is in a position to add, he is constantly misled in the way he is being taught to add. On account of the insistence of the adult to learn the process as a solution, the child is in a position to add.

IV. THE REMOVAL OF DOUBT FROM SITUATIONS

The discussions of the foregoing paragraphs may be summarized in the statement that the creation and perfection of the number system have operated in the direction of removing the element of doubt from the quantitative situations of life. Quantitative situations, which at one time presented themselves as problems — that is, as questions involving doubt — are problems no longer, because the number system has provided clear and well-defined methods of attack upon the situations and has rendered them comprehensible and manageable. The number system has freed the mind from the necessity of dealing with quantitative situations as though they are problems, and thus has provided the opportunity for the mind to move out to new situations that heretofore have been foreign to experience. By providing a method of attack upon the simpler situations of life, the number system has made it possible for the mind to deal with more complex situations.

Illustrations may be drawn from the situations that provided some of the problems of earlier peoples. In the preceding chapter reference was made to the so-called 'problem of pursuit,' or the rate-time-distance problem. That was a real problem in the days of the Romans. A Roman legion, for example, would have to make a march of a given distance, reckoned in thousands of paces. In what time could the legion be counted upon to make the march and arrive at the designated destination? The answer in many cases was a rough approximation. Not being versed in the formula for computing the time from distance and rate, perhaps not having the formula raised to the level of clear consciousness, or not realizing its general applicability, the Roman officer in charge would give his answer in terms of what he could remember about the time consumed on former marches over different distances. He would approximate in terms of this isolated experience or that one, and his answer

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How long will it take to fill the cistern when water flows in through both pipes? Such was the situation in earlier times, but because there existed in the minds of earlier people no well-defined method of dealing with the situation, the situation created a problem. From their previous experiences with the flow of water into cisterns they could make one or more shrewd guesses for their answer, but in any case they had no positive assurance that their guess was correct until they checked their guess by trial and experiment. Moreover, if the conditions of the situation, such as the rate of flow in one pipe or the other, changed ever so slightly during the period of trial, they were thrown into still greater confusion about the guessed answers. Today the city engineer who has the facts about the city reservoir, the supply of water, the average daily consumption, etc., faces no such problem. Though the conditions are a hundred times more complex and on a scale a million times as great, no real problem exists, because the method of attack upon the conditions of the situation have removed the elements of doubt.

V. THE CHARACTERISTICS OF A PROBLEM

What we have been saying may be stated in another way. It is not the situation, nor the various conditions and elements of the situation, that creates a problem; nor is it even the necessity of dealing with the situation that creates a problem. The problem arises out of doubt about the way the situation must be handled. It is ignorance, or uncertainty, about the way the conditions should be dealt with that is the problem or that gives the situation the character of a problem. When this ignorance is removed, when assurance is substituted for uncertainty, when knowledge displaces doubt, the problem as such ceases to exist. This is only another way of saying that whether a problem exists depends upon the degree of intelligence possessed by the individual.

The same situation with identical conditions may confront

two individuals. It follows from what I have just said that to one the situation may present a problem and to the other no problem may be present. Two merchants, belonging to the same country and to the same city, may decide to compete their goods at the same selling price instead of at a different price. Each has decided to fix the price and to realize a gain, which takes into account overhead expenses and is to be 40% of the selling price. In that case, the selling price of an article does not make any difference, a mere coincidence of affairs. Therefore, two merchants. To one it is a problem, to the other it is not a problem. How can the matter be explained — in our case, a problem.

of dealing with the situation. He is disturbed by no doubt as to what should be done. All that confronts him is the task of doing a little subtracting and a little dividing. He can proceed with confidence from the first step to the last.

Both merchants proceed to the same results, but by pursuing different routes. The one solves a problem; the other performs a task. The one is confronted by doubt; the other proceeds to his task with confidence. The one expends mental energy; the other conserves mental energy for situations that are more involved.

VI. THE PURPOSE OF ARITHMETIC IS NOT TEACHING HOW TO SOLVE PROBLEMS

The purpose of instruction in arithmetic may be stated in terms of the service it renders. It has provided the race and it provides the individual with a method of attack upon the quantitative situations of life. It has removed, and it is capable of removing, the element of doubt and uncertainty from such situations. It has freed the mind, and it can continue to free the mind, from the necessity of dealing with such situations as though they were problems. It has operated to conserve, and for the individual it continues to operate to conserve, mental energy for more complicated and more involved tasks. The purpose of instruction in arithmetic is not to teach children how to solve problems; the purpose is to provide them with methods of thinking, with ideas of procedure, with meanings inherent in number relations, with general principles of combination and arrangement, in order that the quantitative situations of life may be handled intelligently and without doubt and uncertainty. The purpose is so to order and systematize the child's methods of dealing with combination and arrangement of objects that he may go through life freed from the necessity of confronting problems of an arithmetical nature. The purpose rests upon the assumption that the individual has a higher function to perform in life than to expend his energy

be chosen a doubtful one. The school can, if it wishes, keep the pupil in doubt about the meaning and uses of percentage; about the development, relations, and uses of weights and measures; and about the characteristics of the unit of square measure; and it will succeed in keeping the situations of charging interest for the use of money, of the computation of gain and loss, of denominate numbers, of determining areas, and so on, on the level of problems to be solved. Whether the school is ever justified in withholding information and training for the sake of antiquated methods of procedure is certainly seriously to be questioned.

It would seem to be the part of wisdom for the school to give the child training in counting before it sets the task of determining the size of a group; to teach the meaning and procedure of division before it expects the pupil to deal with a situation that requires the use of division; and to develop his understanding of percentage, weights and measures, square measure, and so on before it brings to his attention the varied situations of life that employ these ideas and methods of procedure. In order to live in civilized society as a useful member of the social group, the individual must be introduced to the devices and machines and procedures of civilized society, and this before he is called upon and expected to assume the responsibilities that fall to his lot. Shall the young driver of an automobile be allowed to face the situation of determining the side of the road to travel upon, courtesies of the road, and the like as problems to be solved? Shall he be permitted to drive until the need for rules of the road are impressed upon him by experience? Or shall he be taught the rules of the road from the beginning? If a drayman buys a new truck, shall he proceed to use it for hauling while he is still partially ignorant of its operation, or shall he inform himself of the way it should be handled before he undertakes to use it, in order that he may face no problem of operation while he is engaged in the serious conduct of his business?

OLD-FASHIONED 老式風情 咖啡館 位於中區 小坡

The name 'problem' has rather definite designation of the situation that is to be solved, but that has been again a problem. When our students of everyone understands just what type of problem is mentioned. Suddenly to change the nature of the problem would cause misunderstanding, especially those who have learned to use the old name. It is not the name that is the problem, but the incorrect manner of viewing the problem of the teacher.

VIII. Old-Fashioned versus Modern Methods of Investigation

The discussions of the present day of the various
ing chapter have called attention to the various
trasting methods of instruction, in which the
is often emphasized by the various ideas
They are often not only the various
names 'old-fashioned' and 'modern'.
method shows the pupils how to
arithmetic; that is, sometimes the
simple and complete operations
pils how to 'solve problems'.
sents quantitative situations and
by devices ways leads through the
operations used in the present
method shows procedures for the
undertakes in developing a finding
before the pupils are permitted to
procedure. The former method
at least, no 'moral' and
mation for the sake of
The former method is a
the latter works in
'needs' and what is finally

From another point of view, however, the two contrasting methods are very much alike. Both emphasize 'problems' in arithmetic to be solved, the one by showing the method and the other by withholding the method. Both emphasize the mechanical operations of arithmetic, the one by isolated drills and the other by presenting them as ways of getting answers to 'problems.' Both neglect the development of general ideas in arithmetic, the one by emphasizing the various processes as separate and unrelated things to do and the other by emphasizing the various processes as isolated methods of meeting children's needs which are sensed, if at all, as isolated needs. Both interpret arithmetic as a set of tools, each of which may be picked up and used for a particular purpose without any relation to the rest, and both methods show no interest in helping the pupil to fit together the various parts of arithmetic into a smooth-running machine of thinking.

Over against the two methods just described, our discussions suggest that arithmetic ought to become for the pupil a method of thinking that makes understandable the quantitative situations of life. The idea that he should understand what he is called upon to learn by studying the characteristics of the things themselves has been suggested. In this connection it may be well to insert a word of caution about the intent of the discussion of 'problems.'

The insistence that arithmetic ought not to be presented as a series of problems to be solved is not intended to carry the suggestion that the pupil is not to understand what he is learning. It merely suggests that there may be a way of getting him to understand a new step in arithmetic other than the method of presenting it as a means of solving a new kind of 'problem.' What has been said about arithmetic as a system, as a unified body of ideas, as a science, as a means of grouping by tens, and the like, should, perhaps, carry the suggestion that what the pupil has learned and understood about the steps already taken may be the

of confidence in the power and certainty and dependability of his own mental reactions.

Another criticism of the explanations of the teacher (and of the textbook) arises out of the fact that the occasion for each explanation is the pupil's difficulty and unfamiliarity with each new process as he comes to it. To the pupil, each new process usually appears entirely new. It thus appears that the new process must be explained in terms of its newness. As a consequence, each new process, which at first seems so different, is given a new and different explanation. The teacher and the book thus proceed to provide different explanations for the different processes, forgetful of the fact that the very newness of the explanation often renders it quite as difficult for the immature mind to grasp as the process that is being explained.

The trouble with new explanations for new processes is that the pupil is not encouraged to carry over from any one explanation or process any idea that may help him to understand the new process as he comes to it. The pupil is unable to see any connecting link between the different explanations or between the different processes. Each explanation appears to him just as new and strange as the process that seems to need explaining, because the explanation comes too abruptly and without any connection in the pupil's thinking with previous ones.

Worse yet, without any connection being made in the pupil's thinking between the various explanations that are given, they often appear to him as contradictory. A case in point is the explanation that the column to the right must be unbroken when adding whole numbers, and that the column to the left must be unbroken when adding decimals. Both are at fault because they are based upon what the pupil can see with his eyes rather than upon what ideas of relationship may be developing in his mind. Another illustration of contradiction is the explanations that are given of the apparently diminished result when one divides by a

a glimmering of these ideas at the beginning helps the pupil to recognize and rely upon them in the adding and subtracting of two-place numbers. 'Carrying' tens in addition, subtraction, and multiplication, and division by two-place numbers give opportunity for employing the ideas. At every step, to and throughout decimals and percentage, these same ideas continue to appear. Every step may provide the occasion both of illustrating and extending these ideas in the pupil's mind, and of giving him an opportunity to use what little he may previously have learned of them as an aid in his attack upon new processes to be learned -- processes that at first may seem new, but that he may quickly find to be very much like those he is familiar with.

Related to the ideas of ten and position --- really a part of them --- are the ideas of *size* and of *number* (whether of parts or of groups). Although these may be most evident in fractions, they may be understood the better if one can discover them running through the whole of arithmetic. And fundamental to the whole of arithmetic are the ideas of *combination of unequal and equal groups* in addition and multiplication, respectively, and of *separation into unequal and equal groups* in subtraction and division, respectively.

The various general ideas that may thus develop are called by different names and may be distinguished as slightly different, each from the other. On the other hand, they all are closely interrelated, and they are in reality the same general idea of grouping by tens seen from different angles. A glimmering of these ideas, or of the fundamental idea of the group of ten, may be had from the beginning. Every stage of the work may give the pupil increasing opportunity to extend them and to use them in making clear what may seem as new. They may serve to unify what otherwise is a thousand-and-one separate things to be remembered. They may help the pupil to develop a growing feeling of familiarity with what he is learning and with what is set before him to learn, and a growing feeling of confidence and cer-

clarity in the methods of attack. The student may learn to learn. The constant repetition of the idea of ten, for example, and confidence and certainty about the use of any more skill the pupil may obtain may secure by rehearsing these central ideas of arithmetic. The grasp is just so much and on which the thinker. Once a pupil grasps a concept he may be expected to grasp and use it independently of the class or teacher. The idea is an inner drive that gives him power to think for himself. A dynamic factor of experience back upon the experience itself, give larger meanings to these experiences.

XI Development of Mathematical Thinking

The importance of developing mathematical thinking may be illustrated in arithmetic. In the early stages of arithmetic, the pupil encounters problems of the kind of problems. He is asked to find the sum of a number and a number; and a third situation is presented when a group of objects is divided into two groups. He meets the same three kinds of situations. At first, he counts the objects to find the sum. Then, he counts the objects to find the number when a group is divided. Finally, he counts the objects to find the number when a group is divided. As he progresses, he will come to understand that the sum of two numbers is the same as the sum of the same two numbers. He will also understand that the number when a group is divided is the same as the number when a group is divided. Finally, he will understand that the number when a group is divided is the same as the number when a group is divided.

kinds in decimals and still more in their study later in percentage. As the pupil moves from one chapter in his arithmetic to the next, he is aided in handling the new chapter on a higher level of understanding by the use of what he learned about the same situations in the preceding chapter. He approaches the study of the new chapter with some ideas about the new chapter already formed and ready for use. A general method of attack is thus gradually developed.

Moreover, in the later stages of arithmetic, the pupil is required to study what are called the 'applications of percentage': interest, savings, investments, gain and loss, cost and selling price, insurance, taxes, etc. Each topic is new and unknown before it is studied. Each is in some degree a separate topic. Each may continue to appear as something entirely new and different. But, if the pupil approaches the study of each topic with an understanding of the "three kinds of problems" and of the relations and distinctions between them, he may be led to view the various apparently different topics, like interest, savings, and investments, each as further illustration of the same "three kinds of problems" he already knows about. Thus, in the study of interest, the pupil will learn about certain business practices previously unknown to him, and, what is more to the point, he will at the same time gain a better understanding of the old and familiar "three kinds of problems." Thus his general ideas will develop along with the gaining of new information, and thus his preparation for the study of succeeding topics will be strengthened. Such a procedure in studying brings into a single, unified scheme of thinking or method of attack what otherwise might easily be a dozen separate, distinct, and unrelated 'applications' of percentage.

Similarly, in the study of weights and measures, each new topic may appear as something new and different to be learned and as presenting some new information to be remembered; or each may be studied as a means of contributing to the pupil's general understanding of how people

NATURE OF FORMALIST PROBLEM 3

have proceeded in various ways to make things through the last important measure. The study of each of these contribute the share to the general culture and may serve to form the idea that the thing is done as it is standard of contemporary work. The approach to each new subject in the idea of procedure that governs the rules, and thus the learning of a procedure that is to be followed in the study have made clear.

XII The Korean Language

Though has been much said of the artistic and almost the idea and methods of the of the exercise in writing to be revealing and within through enlightenment him, the ing with generalization own. To this end analysis, synthesis of illustration the situation, partially, times with the and the book are a "problem," But it is not a does not leave light upon the procedure, and the

the central theme. Sometimes, as has been indicated, the teacher or book gives the illustration in complete detail, that is, the 'problem' is 'worked.' This is not to show the pupil 'how to work' the 'problem,' but to leave nothing of the illustration to chance. More frequently, however, the teacher or the book gives only a part of the illustration, leaving the pupil to supply the rest of it; that is, to 'work the problem.' And occasionally, the pupil is permitted to provide from his own experience, either direct or vicarious, the complete illustration. In any case, the 'problem' may serve to illustrate something that is of greater importance than itself; and it may serve with other types of illustrations to make clearer the topic that is being studied. *In sum, the 'problem' in arithmetic is in reality not a problem at all, but a type of illustrative exercise.*

CHAP. III. THE

NUMBER AS A THEORETICAL CONCEPT
AS A PRACTICAL AID

此項工程，係由本局委託中國工程顧問公司設計，現已設計完竣，正由本局辦理招標事宜，預計明年即可動工興建。

The following information is for your reference only. It is not intended to be used as a basis for any action.

[illegible]

By the fact that the percentage of persons who are not interested in the subject of international relations is small.

1. I will have access to the information and data of the
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 pit.

6. The purpose of communication is to describe the project in a way that is

Throughout the development of the curriculum, there is to be noticed the emphasis on the relation between what elementary thinking has reached and the perfect science and the unbridled mathematics to be frequently taught and transferred to the other hand, number as a science to be generalized and the other hand, number as a generalized and the role of theory in the development of the curriculum. The curriculum is to be developed in such a way that the elementary and the higher mathematics that are to follow, since they have of the present is providing systematic education to the people.

I. THE SCIENCE OF NUMBER

Through long ages of experimentation the race has perfected the number system that is in common use today. In the beginning, number ideas were the direct outgrowth of perceptual experiences, and as a consequence were meager in scope and relatively vague and indefinite. As time passed, systematic methods of giving attention to groups were created, and the substitution of these methods for the earlier accidental ones led to number ideas that were substantially exact and definite. These ideas were developed concurrently with the creation and development of number names, and they were made serviceable through the use of number names. In turn, the number names both suggested and made possible newer combinations of ideas that were farther and farther removed from direct experience. More and more, attention was withdrawn from perceptual discrimination and centered upon methods of combination. As a result, involved and unsystematic methods were gradually displaced by simple and systematic methods. The Hindu-Arabic system of notation has gradually displaced all other systems in civilized societies.

The Hindu-Arabic system is a decimal system. All numbers beyond nine are expressed in tens and in powers of ten. The idea of ten is the standard by which all chance groups and arrangements are evaluated. The necessity for expressing all quantities beyond nine in tens and powers of ten gradually impresses upon the mind of the individual the idea of ten as a standard and enforces the decimal arrangement as a method of thinking. The individual who learns the system quickly learns to think in terms of ten and to translate all chance groups and arrangements into tens.

The perfection of the Hindu-Arabic system of notation with its uniform standard for the expression and translation of all groups has made possible the discovery of many relations between numbers that hitherto had not been con-

most fundamental of the sciences and to one of the most outstanding intellectual achievements of the human race.

II. THE FAILURES OF PUPILS

Whenever the teacher begins to contemplate the beauty and the simplicity of the number system and the comprehensiveness and utility of its methods of procedure, he is brought sharply to a halt by a realization of the widespread and discouraging failures of pupils in the subject of arithmetic. To many pupils, this subject is baffling in the extreme. Instead of being simple and understandable, it often is bewildering and incomprehensible. Instead of helping pupils to gain methods of thinking and modes of attack upon the quantitative situations of life, it often appears as a means of inhibiting thinking and of beclouding quantitative issues. In spite of longer terms, better buildings, better equipment, and presumably better courses of study and better teachers, the subject of arithmetic remains as the rock upon which the hopes and aspirations of teachers and pupils continue to be wrecked. Not only do pupils fail in arithmetic; their failures are cumulative. Year by year pupils continue to fail, and the failures of given pupils become more and more serious. The two chapters immediately preceding have indicated a possible explanation of such failures. They deserve our further attention, however.

III. UNSUCCESSFUL ANALYSIS

A century and a quarter ago Pestalozzi was engaged in the task of organizing instruction on the principle of proceeding from the simple to the complex. His plan was to divide and subdivide a subject into its minutest parts, and to teach these 'simple' parts to his pupils one by one. Thus, in reading, his pupils were expected first to learn letters, then syllables, then words, and, finally, phrases and sentences. In writing, his pupils had first to learn to make

straight lines, curves, areas, ellipses, and by now and thereby they are able to combine these various parts and unities into wholes.

We have moved far from the piecemeal type of mathematics advocated by Pestalozzi in our teaching of teaching and learning. We remain, however, in the Pestalozzian period of meticulous dissection in our teaching of mathematics. In the twentieth century, our arithmetic is in the same state as the early nineteenth-century period of development. Thus each subject has been analyzed for the pupil into a multitude of combinations, processes, formulas, rules, types of problems, etc., and the pupil is taught each in turn as a separate item of experience. Often, when he has completed the course, he knows only those parts that he can still remember, and they all seem to him as separate and unrelated considerations, processes, formulas, rules, and types of problems to be solved. Finally, when his memory for these separate items fails him, he has nothing left to carry with him except a few but the remembrances of a series of disconnected, uninteresting, and unpleasant experiences that his teacher or arithmetic seem to have provided for him.

Successful teaching in these subjects is what mathematics results are evident, and the success of some pupils in gaining an understanding of arithmetic as a unified system of ideas and modes of attack often in spite of the efforts of the school, furnish the strongest possible arguments against the time-worn method of artificially breaking a subject into parts and trying to teach the parts piecemeal and without logical and understandable relations. Instead of an artificial analysis, with haphazard methods of trying to take the pupil from parts to wholes -- indeed, often of trying actually to teach parts without any relation to wholes -- the correct procedure is that of proceeding from wholes to parts or related wholes. Analysis is always necessary in study. Synthesis, which may build the parts into a unified and understandable unity, is of paramount importance.

IV. THE RESORT TO DRILL

The present piecemeal organization of arithmetic leaves the pupil with no ideas of how to proceed except those that he may gain through verbatim memorization of directions and the blind following of formula and rule. There is little left for him to do but to follow the directions set for his guidance, and to practice following them until they stick in his memory. Moreover, there is little left for the teacher to undertake but to conduct the practice periods as effectively and as expeditiously as possible. Thus, drill has come to be the type of classroom procedure that is peculiar to, and characteristic of, arithmetic. Drill has come to assume an importance far in excess of its merits.

The importance of drill, the actual as well as the assumed, has led many writers and teachers to confuse drill with instruction, instead of distinguishing between the two, and to organize their teaching procedures as drill procedures. As a consequence, many pupils have failed to *comprehend* the combinations and processes they have mechanized through practice, even though they have mechanized the correct procedures; and many other pupils have mechanized round-about and ineffectual procedures that extra amounts of drill have failed to correct.

Reasonable accuracy and reasonable speed, the standards for which have been set through numerous studies, are the proper goals of drill. But because drill has been confused with instruction, accuracy and speed are set down as the real goals of the school course in arithmetic by the textbooks and teachers' manuals in common use. When instruction is distinguished from drill, however, and kept in mind as the first step in classroom procedure, the pupil's understanding of processes becomes the primary goal of the school course, to which may be added the two goals already named. Arithmetic now becomes what the race has perfected and passed along and what society obligates the school to develop;

CHAPTER X

THE STUDY OF GROUPS

ARGUMENT

1. The idea of number is the idea of the group. It is developed and clarified through the study of groups. Its nature is determined by the kind of study pursued, whether haphazard or systematic, incidental or planned.

2. The chapter describes the following methods of studying groups to ten:

- (a) Counting groups;
- (b) Comparing groups;
- (c) Taking groups apart and putting the parts together;
- (d) Emphasizing equal groups.

3. The general procedure suggested is as follows:

- (a) Teach the method of study;
- (b) See that pupils use the method.

4. Learning to count includes

- (a) Learning the number names in serial order;
- (b) Learning to discriminate objects;
- (c) Learning to use the number names as a means of discrimination;
- (d) Learning to give attention to the group as a whole.

5. The comparison of two groups involves at the outset: (a) counting both groups, (b) matching the two groups one-to-one, (c) counting the excess of one over the other. Later, the process may be chiefly that of counting the excess.

6. The language to be learned and used, both the oral and the written, is to express the comparisons the pupils make and understand. The language is not a substitute for ideas.

7. When groups are studied through (a) analysis and (b) synthesis, the pupils are to determine the answers by means of counting. The purpose is to enlarge ideas of groups. Learn-

1. The language of the constitution is to be interpreted in its plain and ordinary meaning, and not in a technical or legal sense.

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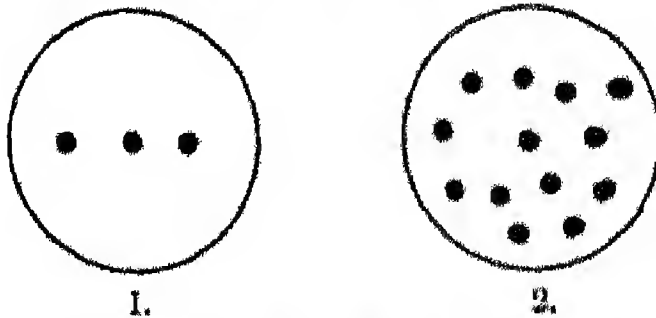
13. The language of the constitution is to be interpreted in its plain and ordinary meaning, and not in a technical or legal sense.

14. The language of the constitution is to be interpreted in its plain and ordinary meaning, and not in a technical or legal sense.

I. INCIDENTAL VERSUS SYSTEMATIC STUDY

The study of groups may be incidental, haphazard of the sort that accompanies one's ordinary, everyday experiences with groups of objects, or it may be an organized, systematic study. Both types of study lead to the development of number ideas — the former, to ideas that are inexact, vague, and indefinite; the latter, to ideas that are exact, clear, and definite. Let us illustrate.

In each circle, numbered 1 and 2, is a group of objects. One glance at the objects in Circle 1 is sufficient to determine the number of objects in the group.



and to be certain of the fact that there are three objects in the group. One glance at the objects in Circle 2 is sufficient to discover that there are several objects in the group. The adult merely looks at the second group of objects; his eyes upon first one, then another object, study them from various angles, and all the conclusion he can reach is that the number is 'more' than the first group — 'quite a few,' 'a lot,' 'many.' He cannot tell exactly how many unless he takes the time to deal with the group systematically. To determine exactly the number in the second group — that is, to become fully acquainted with the group — one must count (or deal with the group in some other systematic way). Perhaps he can recognize a group of three, a group of four, and a group of five, and then combine these three groups together. Whatever he does, he realizes that it takes much more systematic effort to apprehend

gained by a contemplation of the objects of a group, this difference is perhaps in favor of the young child. Experience with the group is, of course, necessary if one is to determine the number. But mere experience is not productive of the number idea as the adult knows and uses it. The experience one gains from his contemplation of the group must be organized experience. The method of organizing experience is the possession of the adult. The young child must be brought into possession of the adult's method if he is to be expected to gain number ideas which have the qualities of definiteness and exactness.

III. METHODS OF STUDYING GROUPS

It has been pointed out that, if the adult cannot at once give full attention both to the group as a whole and to each and every member of it, he carries through the double process piecemeal and step by step. He either counts — attends to each member one by one, and finally all together — or breaks down the whole group into smaller groups and then thinks the smaller groups together. His method of study is the combined method of analysis and synthesis; in one way or another, he takes the group apart, then puts it together. The fact that the double process may be entirely a matter of thought does not make it any the less an actual process.

Groups are studied just like anything else. To get an idea of a machine, of a situation, of a problem, one must first break it down into its component parts and then put the parts together in their proper relations. One may learn about a clock, for example, just by looking at the clock as a whole. One may learn much more about it, about what makes it run, about its reliability as a timepiece, and so on, if he can learn how to take it apart properly — that is, systematically and in an orderly way — and finally, how to put all the parts back together systematically and in an orderly way.

This chapter will mention certain methods of studying

group, each with its own ideas. Each member of the group is free to the whole group, and the group leads to the development of ideas.

Common to all of these is the fact that from certain sources of knowledge, studying averages and the group usually has the right to know only to the extent of what is known. Brownell has pointed out the fact that the group is a too-hasty generalization, a 'herd' to 'abandon' a tradition, the process of the group is that process of 'herd' as a systematic work.

The following are the causes: (1) The group spirit and the group planning equal results.

a. Should the group be in the room, the group should teach the group, frequently mentioned in children in the group, no reason to believe.

- W. A. Brownell, *The Development of the Primary Grades* (Chicago, Chicago, 1924)
- C. H. Judd, *Psychological Foundations of Arithmetic* (Department of Education, 1927)
- G. T. Brownell, *Education, The University of Chicago Press*

ing when they add because they have had *too much* training in counting. They count because they have not learned better methods of adding. It will be pointed out in later pages that addition is a step in advance of counting, and that an understanding of the process of addition is based upon an understanding of processes that are simpler. One should not argue that the child should learn to walk without practice in crawling. The fact that short steps must be learned before longer steps can be taken does not argue that the longer steps should not be thoroughly mastered, nor does the importance of the longer steps argue for a neglect in the training in the shorter ones.

Our first-grade work in number, accordingly, is begun with some training to count and with practice in counting.

b. The Process of Counting. Beginners make mistakes in counting. Four of them may be cited.

1. Sometimes the number name is forgotten, or the names are used in the wrong order. The child counts, *one, two, three, four, five, six, eight, nine, ten, twelve, eleven.*

2. Sometimes the child fails to recognize the point where he has looked and so becomes confused. The adult often makes this error, especially when the objects are very similar or very close together. The failure here is one of discrimination.

3. Sometimes the number names and the discriminated objects are not properly matched. The eye runs ahead of the voice, or lags behind, and confusion results. When one counts the cars in a procession, for example, he does not merely say the number names, nor does he merely look at, or point to, or nod to, each individual car. He must do both, and he must do both at once. If he is careless in distinguishing the cars, he does not know when to cease giving the number names. If he distinguishes each car, but becomes mixed in giving the names, he counts inaccurately. Giving the number name and discriminating the individual car to which the given name belongs must be done together.

1. 凡屬本會之職員，其職務範圍應由各該機關或團體負責辦理。如有不肖之徒，利用職權，貪污受賄，或有不當行為者，一經發現，即行撤職，並送法究辦。此項規定，旨在維護本會之聲譽及利益，望各會員一體知照。

[illegible]

C. Thompson was 1st of 10 children

[illegible]

2. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

3. Temporary or interim criminal in turn

4. Transition to the next slide.

Real progress in the field of human rights can only be achieved by a comprehensive approach that takes into account the needs and aspirations of all people, regardless of their race, ethnicity, or social status. It is essential to create a society where everyone has the opportunity to live in dignity and peace, free from discrimination and violence. This requires a commitment to justice, equality, and the rule of law, as well as a willingness to listen to the voices of the marginalized and oppressed. Only through such a holistic and inclusive approach can we hope to build a better world for all.

counted — thought of together — in a group. In order to bring out for the child's attention the meaning of the answer when he counts, the teacher must so direct the pupil that he will be required to give attention to *each* object and *all* the objects together in a group. "Count the books" merely tells the child to do a thing that can be outwardly observed. "*How many books are on the table?*" is a question that only counting, with all that it implies, will answer. It is the type of direction that requires the fixing of attention upon a particular group. Do not direct the pupil to *count*, but give him the *question* that demands counting as a means of arriving at the answer.

d. Preliminary Testing. From one-fifth to one-third of the pupils at the beginning of the school year will be unable to give the number names to ten with complete accuracy and assurance. Most pupils will be able to give the names to ten, some far beyond ten. When they count objects, however, they often pay slight attention to the objects and as a consequence make errors of the second and third types named above. The teacher should not assume that the mere ability to give the number names in serial order is the ability to count.

The teacher's first task is to *test each pupil's ability to count*. This testing should be very informal. It may proceed as follows: Ask the children to count as far as they can. Ask them to count the pupils in a row, the books on the table, the chairs at the front of the room, the windows, etc.

As a guide to the types of teaching needed by individual pupils and by the class as a whole, *the abilities and difficulties should be listed*. These should be noted at the time when the pupils are tested, and tabulated in convenient form, as indicated at the top of page 189.

e. Counting to Ten. The preliminary activities of testing the abilities of the pupils give practice in counting to ten to those pupils who already have this ability and some training to the others who have difficulty. The weaker ones learn

Date		Time		Place	
1944	10/10	10:00	11:00	St. Paul	St. Paul
1944	10/11	10:00	11:00	St. Paul	St. Paul
1944	10/12	10:00	11:00	St. Paul	St. Paul
1944	10/13	10:00	11:00	St. Paul	St. Paul
1944	10/14	10:00	11:00	St. Paul	St. Paul
1944	10/15	10:00	11:00	St. Paul	St. Paul
1944	10/16	10:00	11:00	St. Paul	St. Paul
1944	10/17	10:00	11:00	St. Paul	St. Paul
1944	10/18	10:00	11:00	St. Paul	St. Paul
1944	10/19	10:00	11:00	St. Paul	St. Paul
1944	10/20	10:00	11:00	St. Paul	St. Paul
1944	10/21	10:00	11:00	St. Paul	St. Paul
1944	10/22	10:00	11:00	St. Paul	St. Paul
1944	10/23	10:00	11:00	St. Paul	St. Paul
1944	10/24	10:00	11:00	St. Paul	St. Paul
1944	10/25	10:00	11:00	St. Paul	St. Paul
1944	10/26	10:00	11:00	St. Paul	St. Paul
1944	10/27	10:00	11:00	St. Paul	St. Paul
1944	10/28	10:00	11:00	St. Paul	St. Paul
1944	10/29	10:00	11:00	St. Paul	St. Paul
1944	10/30	10:00	11:00	St. Paul	St. Paul
1944	10/31	10:00	11:00	St. Paul	St. Paul

through the...
 repeating...
 training...
 be necessary...
 care in...
 part of the...

f. Creating...
 cision of the...
 ately to the...
 the people...
 number...
 an exercise...
 ing to twenty...
 perfect their...

Be ever the...
 these quickly...
 listens to the...
 learns the...
 and the...
 assist those...

Saying the...
 merely preparative...
 in order to...
 When the...
 should be led...
 tice in saying...
 ing serves...
 begin as soon...

children can learn the names they do not know by listening to and imitating the actual counting of the other children.

Teach discrimination. The teacher must make sure that the children discriminate accurately the objects counted. Actually touching each object is a very positive means of discrimination. Pointing to the objects or holding the hand at the objects, one after the other, are common means. Giving the number name to the object, in counting it, assists to fix the discrimination as soon as it is made. The point for the teacher to keep in mind is that the child must see each object in the group, both as a member of the group and as distinct and separate from the rest of the group.

Give practice in counting. It should be borne in mind that the three types of training in counting that have been mentioned are received in coordination, and not necessarily one after the other. Practice in counting helps to fix the number names, gives practice in discrimination, trains in relating the names to the objects, and develops the idea that the name given to each object counted denotes the size of the group up to and including it. The idea of the group comes not through *telling* the pupil about it, but through the child's own practice in reacting to the group in a systematic way; that is, through *actual counting*.

g. Counting Exercises. In conducting the counting exercises, let the children count any set of objects in the room, pupils, desks, chairs, tables, doors, windows, erasers, blackboards, books, pencils, marks on the boards, etc. Let the children count in concert and individually, the better pupils helping the weaker ones at points of difficulty. Let the children count aloud, then quietly to themselves. When the children have counted a set of objects *to themselves*, if all give the right answer, have them count a larger group. If some give incorrect answers, have these children count aloud, one after the other, until the correct count is made.

When a child gives an incorrect answer, have him count aloud, pointing as he counts, to determine wherein he is at

2. Introduce the symbols 10 to 20 incidentally as the occasion for writing these upon the board requires. The pupils may go as far as they are able in writing these at the beginning. However, practice in recognizing and writing these is to greater effect in connection with the teaching of their significance, as indicated in a later chapter.

3. Do not introduce the zero, 0, as such at this time. Practice in writing it may be given in connection with the writing of 10 and 20.

4. Introduce the symbols out of their serial order. Guard against the chance of suggesting the idea that 5 stands for the fifth object counted, 6 for the sixth object, and so on. As the pupils apprehend a group of six objects through counting them, write the 6 to stand for the group.

5. Give practice in recognizing the symbols. Give such directions: "Point to — boys" (writing the symbol desired upon the board). "Count — books" (writing the symbol desired), etc.

6. Give practice in writing the symbols. Such practice should be governed by the rules that apply to practice in handwriting. Make your own symbols large and plain. Show exactly how each symbol is made, in order that no pupil will begin by making the symbol backwards. Let the pupils make the symbols freely in the air, next on the board, and finally upon their papers. Make special efforts to get pupils to improve their writing by calling attention to possible improvements, such as making the last movement of the 2 a straight line and not curved under like the 3; the proper direction and right size of the loop of the 6; the right direction of the movements in making the 7, so there will be no tendency to make it backwards.

4. *Summary.* When one counts a group of objects, he engages in a systematic study of the group. He gives his attention to each object, one by one. He applies to each a name. The name serves the double purpose, first of setting the object off by itself as a separate entity from all the

Before very long the children will be required to use each of the number ideas up to nine, in combination with the others, in what we call the combinations of addition, subtraction, multiplication, and division. They will be expected to be intelligent in their use of the number ideas in combination, to understand what they do when they add, subtract, and so on, and to have confidence in their ability to carry through the combinations successfully.

In order to use ideas with confidence, certainty, and understanding, a person needs to have the ideas fairly well developed in his mind. In the use of an idea in a given circumstance, one does not need to use all that he knows about it; he may need to use only one part or one phase of the idea. But the more he knows about the idea, the more certain he is in the use that he has to make. If the idea is very familiar, he can use it in this circumstance or that one, with perfect confidence in himself and with complete understanding of what he is trying to do.

Now, as a result of their counting activities, the children know something of the number ideas to *nine* or *ten*. They have started in their thinking the development of the ideas. They know the serial order of the numbers, and that is knowing a lot for beginners. They realize that *eight*, for example, is more than all that go before *eight*, and less than all that follow. They may know *eight* pretty well in its relation to *seven* and *nine*, but they are very vague with respect to the relations of *eight* and the other number ideas. And there is much about the idea *eight* with respect to the various groups that may compose it that they do not yet know. They may use *eight* with perfect confidence in counting, but they would be at a loss how to use *eight* in combination with other ideas. The idea of *eight* is only partly developed. It must now be clarified, and developed, and enlarged. By the use of the process of counting, the children need now, therefore, to be led to a further development of their number ideas.

In a very thorough study of the development of children's

number of objects is not too large
the objects should be of different
materials, colors, shapes, sizes, etc.
the objects should be of different
textures, weights, etc.
the objects should be of different
values, etc.
the objects should be of different
interests, etc.

In the study of objects, the child
he made his own selection of
objects from the group and
have started for himself
called them names, etc.
the children might be asked to
through a sorting exercise.

c. Developing Vocabulary
In the study of objects, the
purpose of the study is to
interested in the objects
objects. The objects should be
for the purpose of learning
or to any of the objects
for the value and interest
may be grouped, etc.
arranged, etc.
it is not possible to
ing by means of the objects
objects in various groups.

The objects should be of
small enough to be used
unusually attractive, etc.

* Broadell, op. cit.
* The objects should be of different
But because the objects are of different
ways it may be studied in different ways.

jects, but with the grouping or arrangement of objects, nothing should be introduced that will call attention away from the essential thing, which is their grouping, or arrangement. Therefore, the less attraction the objects have, the better suited they are to the purpose in using them here.

d. Developing Number Ideas by Comparing Groups of Objects.

Comparison is an important means of developing and clarifying the children's ideas about the sizes of groups. Thinking of one group in comparison with another serves to make both more understandable and usable. What is already known about each of the two helps to extend knowledge of the other. Through comparison the number ideas, which are already partially developed through attention to groups in counting, are seen in clearer perspective and from newer points of view. Through comparison the children are enabled to organize their experiences somewhat differently from what they otherwise would, and thus to enlarge their ideas through the newer form of reacting to their experiences.

In the activities of comparing groups the children are required to give attention to two groups of objects, and thus to the two ideas that attach to the two groups, at once. They cannot give exclusive attention either to the one group or to the other. Attention must move from the one to the other, and back again; rather, attention must fix upon the relation between the two until the relation between the two has been definitely established. A new form of consciousness is developed that is not the same as the consciousness of either the one group or the other under comparison. This new relational form of consciousness is in a sense a new idea; but an idea nevertheless that intimately relates to the ideas that already belong to the two groups. In another sense the new idea of relation becomes a part of, and adds to, the ideas already in mind. Through comparison the children develop both a new idea and a connecting link between ideas that are already partially developed.

*Illustrative Lessons**I*

The teacher indicates a row of pupils sitting at their desks, and asks, "How many children are in this row?" The children count, and answer (seven, let us say).

The teacher holds up five pencils, and asks, "How many pencils have I?" The children count, and answer.

The teacher asks, "Are there more children than pencils?" and, when the children have answered, asks, "How many more?" If the children can all answer, the teacher may summarize, as indicated later. If the children hesitate, the exercise may proceed, as follows:

The teacher says: "There are seven children in this row. I have five pencils. I will pass the pencils to the children." She gives each child a pencil until the pencils are gone. "How many of the children have no pencils?" The pupils can answer.

The exercise is now summarized. The teacher inquires, "How many children?" (Seven) "How many pencils?" (Five) "How many more children than pencils?" (Two) "Which is more, seven or five?" "How much more?" Finally, the teacher provides the statements, if the pupils do not already know how to give them: "Seven is two more than five," and "Five is two less than seven."

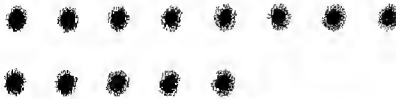
2

Five girls may be lined up in a row at the front of the room and three boys in a row close to them. Ask, "How many girls?" "How many boys?" "How many more girls than boys?" The teacher may suggest the pairing of the three boys with three of the girls, and when they have paired, indicate the two girls left unpaired. If the pupils need the demonstration of actual pairing, it may be carried out; if they can give their answers without it, it does not need to be used.

When it is clear to all the children that there are two more girls than boys, the exercise is to be summarized into the statements, "Five is two more than three," and "Three is two less than five." Likewise, books and pencils may be used to demonstrate one-to-one correspondence. In each book a pencil may be put, and the resulting inequality may be readily observed.

f

Eight dots may be made with white chalk and five dots with red chalk, thus:



The usual question is, "How many white dots?" "How many red dots?" can be asked by the children, because they can count.

The usual questions about "Which is the more?" "Which is the less?" "How many more?" "How many less?" are now given, and answered. Finally, the question is mentioned or introduced above.

Many of the children will suggest pairing up the dots or ten with ten, or five with five, and have many other ways similar to those above mentioned. The teacher may suggest pairing, and in many cases permit the pairing to be made, to help those pupils who have in previous work found answers. When, however, the pupils can give their answers readily without such pairing, they should be permitted to do so. The point to be kept in mind is that the pupils must be permitted to do their own comparing. It is upon these early activities alone that a development of their sense of comparison. The question, "How many more?" is usually mistaken as direct attention to the relation between two groups.

g. The Oral Language of Comparison. Comparison requires a language of its own. Such terms as more and less, larger and smaller, longer and shorter, older and younger, farther and nearer, and the like must be used back to front the pupils' comparisons and to describe the comparisons after they have been made. In the earlier work of counting, some of these terms have been used and they are not entirely unfamiliar to children when they first enter school.

In calling attention to two groups of objects and describing the comparison of them, many questions and answers are

used. At the close of a comparison, when the idea of relative size is perfectly clear to the children, the teacher provides the language that summarizes the idea of relation into a concise statement -- "Five is two more than three," for example. As the work progresses, the children become able to do their own summarizing.

The value of the concise summary statement is due to the fact that it serves to hold in thought a comparison that has been very elaborate and time-consuming in the making. It helps the children to keep in mind what has just been done, and it serves later as a means of making a quick review.

It needs to be emphasized that such summary statements do not precede the thinking and the doing of a comparison. They follow. The idea of comparison must be developed first. Then, and only then, will the summary statement be of value.

Another point to be emphasized is that the summary statement, such as "Five is two more than three," is not given to be memorized at once. The bare statement can be memorized with ease without the children realizing what it is intended to describe. They can memorize words and neglect ideas. In time, as the ideas develop, however, the statements naturally will be memorized. Let meanings be insisted upon, and the memorizing will eventually take care of itself.

h. The Written Language of Comparison. Since the pupils are to learn gradually to depend upon the written language of number to aid them in their thinking, this language should be gradually introduced. Like the oral language, it is not, however, to be introduced in the beginning. It needs to follow the idea, because it should stand in the children's thinking as a way of expressing an idea already possessed by them.

At first the written form should be introduced as a substitute for the oral question. Finally, it may be understood as

The written statement of a confidential source of the FBI dated 1/24/68, through which it was learned that the above information was obtained from the source.

[illegible]

8
 Date: 3. 24. 1944
 is written on 10. 12. 1940

The next question is, "Is it possible to have a five?" She replies, "No, it is not possible." She explains that this is because if you have a five, you are not right about right. She says, "If you have a five, you are not right about right."

Answer the questions 1-2 in Chinese.

Or, the eight? In 5, and 1-7

abc 14 20 26 32 38 44 50 56 62 68 74 80 86 92 98 104 110 116 122 128 134 140 146 152 158 164 170 176 182 188 194 200 206 212 218 224 230 236 242 248 254 260 266 272 278 284 290 296 302 308 314 320 326 332 338 344 350 356 362 368 374 380 386 392 398 404 410 416 422 428 434 440 446 452 458 464 470 476 482 488 494 500 506 512 518 524 530 536 542 548 554 560 566 572 578 584 590 596 602 608 614 620 626 632 638 644 650 656 662 668 674 680 686 692 698 704 710 716 722 728 734 740 746 752 758 764 770 776 782 788 794 800 806 812 818 824 830 836 842 848 854 860 866 872 878 884 890 896 902 908 914 920 926 932 938 944 950 956 962 968 974 980 986 992 998 1004 1010 1016 1022 1028 1034 1040 1046 1052 1058 1064 1070 1076 1082 1088 1094 1100 1106 1112 1118 1124 1130 1136 1142 1148 1154 1160 1166 1172 1178 1184 1190 1196 1202 1208 1214 1220 1226 1232 1238 1244 1250 1256 1262 1268 1274 1280 1286 1292 1298 1304 1310 1316 1322 1328 1334 1340 1346 1352 1358 1364 1370 1376 1382 1388 1394 1400 1406 1412 1418 1424 1430 1436 1442 1448 1454 1460 1466 1472 1478 1484 1490 1496 1502 1508 1514 1520 1526 1532 1538 1544 1550 1556 1562 1568 1574 1580 1586 1592 1598 1604 1610 1616 1622 1628 1634 1640 1646 1652 1658 1664 1670 1676 1682 1688 1694 1700 1706 1712 1718 1724 1730 1736 1742 1748 1754 1760 1766 1772 1778 1784 1790 1796 1802 1808 1814 1820 1826 1832 1838 1844 1850 1856 1862 1868 1874 1880 1886 1892 1898 1904 1910 1916 1922 1928 1934 1940 1946 1952 1958 1964 1970 1976 1982 1988 1994 2000 2006 2012 2018 2024 2030 2036 2042 2048 2054 2060 2066 2072 2078 2084 2090 2096 2102 2108 2114 2120 2126 2132 2138 2144 2150 2156 2162 2168 2174 2180 2186 2192 2198 2204 2210 2216 2222 2228 2234 2240 2246 2252 2258 2264 2270 2276 2282 2288 2294 2300 2306 2312 2318 2324 2330 2336 2342 2348 2354 2360 2366 2372 2378 2384 2390 2396 2402 2408 2414 2420 2426 2432 2438 2444 2450 2456 2462 2468 2474 2480 2486 2492 2498 2504 2510 2516 2522 2528 2534 2540 2546 2552 2558 2564 2570 2576 2582 2588 2594 2600 2606 2612 2618 2624 2630 2636 2642 2648 2654 2660 2666 2672 2678 2684 2690 2696 2702 2708 2714 2720 2726 2732 2738 2744 2750 2756 2762 2768 2774 2780 2786 2792 2798 2804 2810 2816 2822 2828 2834 2840 2846 2852 2858 2864 2870 2876 2882 2888 2894 2900 2906 2912 2918 2924 2930 2936 2942 2948 2954 2960 2966 2972 2978 2984 2990 2996 3002 3008 3014 3020 3026 3032 3038 3044 3050 3056 3062 3068 3074 3080 3086 3092 3098 3104 3110 3116 3122 3128 3134 3140 3146 3152 3158 3164 3170 3176 3182 3188 3194 3200 3206 3212 3218 3224 3230 3236 3242 3248 3254 3260 3266 3272 3278 3284 3290 3296 3302 3308 3314 3320 3326 3332 3338 3344 3350 3356 3362 3368 3374 3380 3386 3392 3398 3404 3410 3416 3422 3428 3434 3440 3446 3452 3458 3464 3470 3476 3482 3488 3494 3500 3506 3512 3518 3524 3530 3536 3542 3548 3554 3560 3566 3572 3578 3584 3590 3596 3602 3608 3614 3620 3626 3632 3638 3644 3650 3656 3662 3668 3674 3680 3686 3692 3698 3704 3710 3716 3722 3728 3734 3740 3746 3752 3758 3764 3770 3776 3782 3788 3794 3800 3806 3812 3818 3824 3830 3836 3842 3848 3854 3860 3866 3872 3878 3884 3890 3896 3902 3908 3914 3920 3926 3932 3938 3944 3950 3956 3962 3968 3974 3980 3986 3992 3998 4004 4010 4016 4022 4028 4034 4040 4046 4052 4058 4064 4070 4076 4082 4088 4094 4100 4106 4112 4118 4124 4130 4136 4142 4148 4154 4160 4166 4172 4178 4184 4190 4196 4202 4208 4214 4220 4226 4232 4238 4244 4250 4256 4262 4268 4274 4280 4286 4292 4298 4304 4310 4316 4322 4328 4334 4340 4346 4352 4358 4364 4370 4376 4382 4388 4394 4400 4406 4412 4418 4424 4430 4436 4442 4448 4454 4460 4466 4472 4478 4484 4490 4496 4502 4508 4514 4520 4526 4532 4538 4544 4550 4556 4562 4568 4574 4580 4586 4592 4598 4604 4610 4616 4622 4628 4634 4640 4646 4652 4658 4664 4670 4676 4682 4688 4694 4700 4706 4712 4718 4724 4730 4736 4742 4748 4754 4760 4766 4772 4778 4784 4790 4796 4802 4808 4814 4820 4826 4832 4838 4844 4850 4856 4862 4868 4874 4880 4886 4892 4898 4904 4910 4916 4922 4928 4934 4940 4946 4952 4958 4964 4970 4976 4982 4988 4994 5000 5006 5012 5018 5024 5030 5036 5042 5048 5054 5060 5066 5072 5078 5084 5090 5096 5102 5108 5114 5120 5126 5132 51

The whole thing was a big joke.

without saying it. I said, "I don't know," and he said, "You don't know? Well, you don't know how to count. You don't know how to count to 8, or thank God, you don't know how to count to five?" When he said that, I said, "Five is how many fingers?"

are written that
or "How many

The pupils were given the opportunity to find it, the sign for it, or the whole combination of signs, "more than five," or "less than five."

1000

Illustrative Lessons

1

When the children have gained certainty and assurance in their comparison of number ideas through the actual comparison of groups of objects, they should be led to compare the ideas when they relate to things less tangible than objects. This will be a step in advance of what they have been doing, and the work should be attempted gradually, and never forced. For example, refer to actual situations, such as the following:

1. James is 6 years old. His sister, Lucy, is 9 years old. Who is the older? How much older?

2. Willie spent 10 cents for pencils, and 5 cents for candy. How much more did he spend for pencils?

The experiences of the children will contribute many similar situations that will make comparisons possible.

2

In the later stages of the work, when such questions are asked orally as, "Nine is how much more than five?" and in writing as,

$\begin{array}{r} 9 \\ - 5 \\ \hline \end{array}$, the children may be shown how to resort to finger-counting to arrive at the answer. When comparing five and nine, one assumes that he has counted to five, and then counts beyond five to nine, *on his fingers*, thus: *six, seven, eight, nine*; and notes that he has used *four* of his fingers to complete the count.

Of course, this is a very primitive method of making a comparison, and one that the pupil will need to forsake for better methods; but it is a step in advance of any method yet used, because it enables the pupil to *think* the comparison without objects other than fingers, by using his readily accessible fingers in place of the objects. If he never learns a better method, he will continue to use finger-counting, if and when he shall learn a better method, he will forsake his finger-counting. One need not fear the danger of finger-counting becoming habitual of itself; it becomes habitual only to the degree that no better method is learned. Provision will be made for the learning of better methods. Later, the pupil's thinking will be so short-cut and direct that he will have no need of finger-counting. Let him proceed *now* from the more concrete steps to finger-counting; let him use finger-counting as much as

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he needs to know; the demands of the subject are prepared by this step in advance of his actual knowledge of the subject.

3

More at each step is required than the previous step. Many more facts are to be learned than in the first step. Some that are more difficult than the first. Some are required by the oral and written questions. "Eight is more than five," and

i. Some Cautions to Observe in the Use of Questions. First, when a series of questions have been asked, the answers have been given and written, the class

$$\begin{array}{r} 8 \\ - 5 \\ \hline 3 \end{array} \quad \begin{array}{r} 9 \\ - 7 \\ \hline 2 \end{array} \quad \begin{array}{r} 7 \\ - 4 \\ \hline 3 \end{array} \quad \begin{array}{r} 6 \\ - 3 \\ \hline 3 \end{array} \quad \begin{array}{r} 5 \\ - 2 \\ \hline 3 \end{array}$$

be sure to erase the answers at the conclusion of the exercise. The pupils are not ready for constructing and reading written statements; rather, they may try to memorize answers in place of understanding them. If these oral and written statements are eventually to be used as a basis for the children in place of the ideas for which they stand, the answers for which they stand must first be thoroughly understood. The pupils are still beginners; they have no previous knowledge of nothing. They must be led slowly and gradually to mastery. The mere remembering of oral and written answers is not mastery. When the children work the given problem the ideas have become so familiar and so well understood that there is no need for them to return to the construction of concrete objects in actual groups or to other means with certainty and intelligence, they will be ready for abstraction. Indeed, if abstraction is practicable at this point, there will be little need for construction, because

will have slowly and gradually perfected itself in connection with the many thought activities that have been engaging the attention of the children.

Second, do not strive for completeness or absolute mastery of comparisons before proceeding to the activities that are suggested in the two sections which follow. In connection with the work suggested in those sections, let the study of comparisons be returned to. All number ideas are related, and all types of work leading to the development of number ideas are related. This section suggests one way to study groups. The next two sections (3 and 4) will suggest other ways. All are intended to lead to an enlargement of the children's ideas of groups. Let the different phases of the study of groups, then, be taken up, not as isolated phases, but as related phases.

In developing mastery of number ideas, the children will not master first one phase then another. Mastery will be gained along all related lines simultaneously and gradually. Each part of the work will help the other. The activities suggested here will help those to be suggested in the next sections. They, in turn, will reflect back upon those suggested here and give them added meaning.

j. Summary. By means of the activities that have just been suggested in Section 2, the children arrive at an understanding of groups that is more complete than any they had as a result of mere counting exercises. The study of groups has led to an enlargement of number ideas to nine. But the children are not yet ready to use the nine ideas in combinations. The activities of the two sections that follow are intended to lead to a further enlargement and development of the ideas to nine.

3. Studying Groups by Taking Apart (Analysis) and Putting Together (Synthesis)

Section 1 of our discussions has suggested how groups may be studied by giving attention one by one to the individual

objects which compose them. Section 2 has suggested how groups may be studied by comparing one group with another. The present section, 3, will continue the discussion by suggesting how groups may be studied by means of analysis into the smaller groups of which each is composed and by means of synthesis of these smaller groups into the original larger ones. The purpose throughout is the same, namely, to lead pupils to enlarge, develop, and clarify their ideas of number up to nine.¹ Since the idea of number is the idea of the group, we must lead the children to a more systematic study of the group.

a. *Number Facts versus Number Ideas* One must look upon the learning of arithmetic as the learning of organized number facts or one may view the process as the development in the mind of the learner of interrelated number ideas. Viewed from the former angle, the learning of arithmetic is little more than systematic memorization. Viewed from the latter angle, the process is that of an active mind that not only learns the various number facts but also understands their relations and seeks applications of them in application. When number ideas are so developed they may be applied by the learner in clarifying other number ideas he possesses and later in developing number ideas of higher order.

The point of view that will be maintained throughout these discussions is that pupils are to be not so much taught that will lead them to develop number ideas. Thus the various so-called number facts are not merely learned by the pupils but rather, developed by them as constitutive parts of their number ideas, and that the development of the number ideas makes possible the development and learning of the associated number facts in their relations. Consequently the effort will be made to suggest the activities that will lead the pupils to think of six, for example, as five and one, four and two, three and three, etc., as two threes, three twos, as two and two, etc.

¹ Really, up to ten. See footnote 1 p. 174.

than seven, etc., and not to separate in the instruction

3 2

3 and 4 merely because the latter has been suggested by

6 6

some as being slightly more difficult than the former.

At every point the instruction is to be organized to the end that ideas will be developed and meanings gained. Learning arithmetic, thus considered, is not memorizing facts and manipulating number symbols on paper or on the board; it is, rather, the building of ideas out of ideas already gained and the acquiring and the relating of meanings; in short, it is an intellectual process. The processes of arithmetic, thus considered, are thought processes, not paper and pencil processes.

b. The Study of a Group of Six. Let the children's study of the arrangement, or grouping, of objects begin with a group of six. Such a group will be small enough for the pupils to deal with at the start, and at the same time large enough to present a challenge to them. Moreover, it leaves the smaller groups of two, three, four, and five for the pupils to deal with somewhat independently as soon as they have become familiar with the method of study.

Select six objects large enough to be visible to the entire class, but not too large to be easily handled. Pencils will serve the purpose.

/////

1. "How many pencils have I here?" The pupils will determine by counting, since up to this point the activity is a continuation of their preliminary counting activities.

2. "I will now take two pencils away from the six and put them over here. Notice that I take two away."

////

//

3. "How many pencils are left here?" The pupils determine.

4. Repeat the arrangement again and again, emphasizing

the facts that there are six pencils at first, and that, when two are taken away, four are left.

5. Call upon the pupils, one by one, to perform the activity, letting each child tell, step by step, what he does. Use the objects used. Use sticks, beads, pieces of cloth, etc., etc.

6. Continue until each child is perfectly familiar with the fact, because he has seen it again and again, that when two is taken from six, four remains.

c. *The Oral Language of Arrangement, or Grouping* When the pupils have observed the arrangement of objects as just described sufficiently often to understand it thoroughly, the teacher supplies the language that describes it. "There are six, six is four." This the teacher suggests as a substitute for the longer, round-about descriptions, questions and answers, that she and the children have been using to guide their attention to the arrangement. This language may now be used by teacher and pupils in further descriptions of the arrangement, while it is being made, and as a means of describing it, after it has been made.

d. *The Written Language of Arrangement* When the repeated process of arrangement has suggested the meaning, and after the oral language, "Two from six is four," has been connected with its meaning, the teacher suggests the

written language: $6 - 2 = 4$. This the teacher suggests as a

scribe an arrangement while it is in the process of being made. In order that the connection that is made, first, between the written form and the meaning, and second, between the written form and the oral explanation of the meaning, the children should be encouraged to serve the writing of the form step by step.

1. "How many books are there?" The teacher shows a figure for six to show how many.

2. "I take two away." The teacher shows a figure for two to show how many.

with this mark (-) before it to show that I have taken two away." The children may be reminded, if they do not

recall it themselves, that the signs, $\overset{6}{-2}$, also ask a question about 'more' or 'less.' Remind them, and at the same time point out and insist, that it also means "Two from six is how many?"

3. Now, taking the question, asked orally, "Two from six is how many?" and asked in writing, $\overset{6}{-2}$, the answer is found by observing, perhaps counting, the group that remains. The figure for the answer (4) is written in its proper

place, as shown: $\overset{6}{-2}$
4

e. *Use the Sign for Subtraction.* The sign for subtraction (-) should be used during the first two grades. There must be something in the written form of subtraction to distinguish it from the form of addition, at least until the pupils have learned the process well enough to describe it by the word 'subtraction.' (Do not use the word 'subtract' in the beginning.) Call attention to the minus sign (-), stating that it means 'to take away,' or that the number shown by the lower figure is to be taken away, and that it means 'from.'

f. *Read Upward.* In the expression $\overset{5}{-2}$, the reading

should be upward. At first glance, this seems to be inconsistent with the manner of its writing, which is downward, and with the order of attending to the groups involved in the subtraction. One first attends to the group of five. Next, one attends to the group of two that is removed. But the removal of two objects is more than a single act of arrangement or a single act of thinking. One not merely takes away two objects; he takes away two objects from the five objects, and he must have the total in mind while removing the portion. The adult can, of course, just as

readily think, "Five, take away two," or "Two from five" but the child, if he starts with the former expression in his thinking, must remind himself, or be reminded, of the negation of the latter. So, if he starts with the latter expression, he may be helped by including the latter, "Five, take away two from five." In other words, the child is concerning the group of two or in thinking its removal, cannot get lost in the process of thinking to the original group.

Moreover, the expression, "Five, take away two, three," is not a sentence; and if we insert the word, the expression is not in good form. If we substitute the word 'how' for 'take away,' we make a good sentence — "Five how two is three" — but we use a word that implies comparison of two groups when no comparison is undertaken.

We may dispose of the matter in another way. The reading of the simple combinations by the pupil is not easy, and is a matter of enough difficulty to require the reading of the expressions as a separate undertaking. The group words 'Five

5

take away two," — 2, for example. That is the problem. Now follows the actual thinking of the process which is something different. He can just as well pronounce "3 is from five is three" as "Five take away two three." Since the reading of the combination is different from thinking it it may be read in the manner that suggests the arithmetic language, namely, "two from five is three."

g. *Further Study of the Group by Putting 5 together* Now that the group of six has been arranged into a group of four and a group of two, the next thing to be done is to call attention to the process of putting the two groups together to form the original group.

1. "How many pencils are here?" //
2. "How many are here?" ////
3. "I want to find out how many are two and four. I will put them together and see." The two groups are brought together. /////

4. "How many are here together?"

5. "Two and four are how many?" If the children have been attending closely to the preceding demonstrations of arrangement, and are wide-awake to the fact that a *group of six* is being studied, they can answer at once. Otherwise, they can count.

6. The teacher presents the oral language to describe the arrangement of *pulling things together*: "Two and four are six." The new, concise statement is now used by both teacher and pupils to summarize the discussions, and in answer to the questions they have been using

7. The written language is now presented, $\frac{2}{4}$. The pres-

entation is made step by step, as indicated above. As the pupils answer the questions about the two groups and about the final group when the two groups are brought together, the teacher writes first the sign for two, next the sign for four, finally the sign for six. In the process of writing, the

teacher points out that the signs, $\frac{2}{4}$, ask, "Two and four are how many?" and that the completion of the writing answers the question, thus: "Two and four are six." Notice that the expressions are written downward and are *read downward*. There are two reasons: (1) reading the *pulling together* expressions downward helps to contrast them with the *taking away* expressions, which are read upward; (2) a slight advantage is claimed for downward adding in column addition.

8. With the objects now in a group of six, the double operations of *taking away* and *pulling together* are reviewed:

$$\begin{array}{r} \text{"Two from six is four."} \quad \frac{6}{-2} \\ \hline 4 \end{array}$$

$$\begin{array}{r} \text{"Two and four are six."} \quad \frac{2}{4} \\ \hline 6 \end{array}$$

The purpose of reviewing the two operations together is both to bring them together in their relations and to draw attention upon their contrasts.

h. Continuing the Study of the Group of Six The study of the group of six by means of analysis into smaller groups and of putting these smaller groups together into the original should continue. As the work progresses, the teacher may find that the demonstrations of the taking away with putting together exercises can be less elaborate than at first. However, the children with some direction can carry through the necessary arrangements.

The arrangements of the group of six are now studied that lead to an understanding of the following and their reverse descriptions:

"Four from six is two."	6 4 2
"Four and two are six."	4 2 6
"Three from six is three."	6 3 3
"Three and three are six."	3 3 6
"Five from six is one."	6 5 1
"Five and one are six."	5 1 6
"One from six is five."	6 1 5
"One and five are six."	1 5 6

i. Developing Ideas before Using Words and Signs. It will bear repeating that the foregoing oral and written descriptions should follow, not precede, the arrangements of the group of six they describe. The adult mind, in its number-thinking, has moved so far away from attention to concrete groupings, and has used the oral and written descriptions in place of the concrete groupings for so long, that it frequently mistakes the description for the thing it is intended to describe. Thus, the adult refers to a set of symbols, such as

2
 $\frac{4}{6}$, as a 'combination.' For all practical purposes of the

adult the reference is accurate enough. The symbols stand so completely for the combination they record, and the thinking of the adult is so far removed from the need of concrete demonstration, that they may with all propriety be called the combination. But in dealing with children who are in the midst of learning about the combinations, or arrangements, or groupings, of objects, the adult mind must bring itself to the level of the children's minds, and refrain from calling the sign for the thing the thing itself. Actually,

2
 the symbols, $\frac{4}{6}$, are not the combination, and the oral sen-

tence, "Two and four are six," is not the combination. The combination is the actual process of combining, of the bringing together of a group of two and a group of four into a group of six, or the counting of a group of two and a group of four together in such a way as to discover that both together are six. *The symbols and the sentence describe what has been done.* The thing is first done, then talked about and written about. At the start the descriptions of the combinations, or arrangements, cannot properly be used as a substitute for the arrangements themselves. Later, when the children have acquired mastery through many repetitions of the actual arrangements, they may themselves substitute the oral and written language for the combinations

the language describes. It must be borne in mind that such substitution must be made by the developing records of the children, and that, when it comes, it must come as a result of gradual mental development.

So, let the teacher refrain from speaking of the square,

2

4, and the like, as the combinations and relations from which

6

ing that these signs, as "combinations," for instance, of children are so prone to memorize the oral and written combinations of the combinations, instead of giving attention to the meaning combinations, that the teacher should encourage, and encourage, them thus to memorize. Let the teacher be reminded that the children are studying groups, and that his oral and written statements of which we are speaking are useful only as they help the children to understand and to keep in mind the various steps they actually take in the study of groups.

j. *Progress of Pupils in Studying Groups* In the beginning pupils are entirely dependent on their study of groups upon the directions and suggestions of the teacher. They must gain independence. They must be able to study, and gradually upon their own motion. From the beginning the teacher must plan to give them freedom in doing things more and more for themselves. Moreover from the beginning, the pupils should be expected to make progress in a slow and gradual program, to be sure, in their study, to give attention to the arrangement of objects and to give so much attention to the objects themselves.

Since the idea of number is the idea of the group, and with the idea of the objects, objects are studied in the beginning in order to provide an actual group for study. As the pupils become able, through study and practice to give more and more attention to the group and to the arrangement of the group, objects may be used less and less. Finally the objects may be dispensed with entirely.

The various steps of progress in the ability of pupils to give attention to arrangement are indicated in the following outline:

1. Attention to arrangement of objects when the teacher makes and describes the arrangement. This step has been described in the paragraphs dealing with a study of the group of six.

2. Attention to arrangement of objects when the pupils make and describe the arrangement.

3. Attention to arrangement of objects when the pupils do not actually make, but *think*, the arrangements, and then describe them.

4. Attention to arrangement when the objects are present only in imagination.

5. Attention to arrangement when no objects are present, employing the language of number to describe the arrangement, which is now entirely a matter of thought.

We shall proceed to describe steps 2 and 3, reserving for a succeeding chapter a description of steps 4 and 5.

k. Independent Work. Step 2 may be described as follows: Following the study of a group of six, as outlined in preceding topics, let each pupil have a group of six objects to study. The teacher directs the study by asking such questions, orally and in writing, as the following:

$$\begin{array}{r} \text{"Two from six is how many?" } \quad \begin{array}{r} 6 \\ - 2 \\ \hline 4 \end{array} \\ \text{"Two and four are how many?" } \quad \begin{array}{r} 2 \\ + 4 \\ \hline 6 \end{array}, \text{ etc.} \end{array}$$

Each pupil arranges the objects in his group of six so as to demonstrate the arrangement called for and to determine the answer. When he has determined his answer, not merely remembered it, or had it suggested to him by another pupil, he may give his answer orally, or in writing, or in both ways, as required.

When the pupils are all familiar with the method of determining the answers to the teacher's questions about the

group of six, they may next be given a group of five, and permitted to determine answers to the same questions.

The teacher asks, for example:

"Two from five is how many?" 3
 "Two and three are how many?" 5

Each pupil determines his own answer and gives it as requested.

In a similar manner the pupils proceed gradually to a study of a group of seven, of eight, and of nine.

1. Thinking the Arrangement. As the pupils enter the classroom in the second step of their study of groups, they will develop a growing feeling of familiarity with what they are to study and with what is expected of them. Therefore, before they begin to become so familiar with the groups, the teacher should begin to encourage them to think the arrangement. They can find and give their answers to the questions without going to the trouble of actually making the arrangements suggested. For example, when the question "If we have five, how many?" is given, they may be able to give the answer five, by thinking, perhaps, something like this: "Five is five, without actually bringing them together." They should be encouraged to make ahead for the study of the groups together and thinking things out.

The familiar dot arrangement of numbers is used in the study of groups (dots) as a means of helping the pupils to think the arrangement. When they have learned to think of the dots, they have no trouble in giving the answer. The teacher presents the diagram shown below:



and asks, "How many dots are here?" and asks, "How many dots are here?" and asks, "How many dots are here?" and asks, "How many dots are here?" and asks, "How many dots are here?"

When the pupils are ready for step 3, they may be instructed in the use of marks or dots on their paper or on the board. Thus, in studying the group of seven, the pupils may use seven marks on their paper

/ / / / / / /

In answering the question, "Three from seven is how many?" they may think the marks away by covering three of them with a finger and counting the number left.

m. Reviewing the Study of a Group. After a group has been studied, it should be returned to in frequent reviews. At first, the pupils answer the teacher's questions, "Four

$\frac{4}{6}$ and two are how many?" $\frac{2}{6}$, "Five from six is how many?"

$\frac{-5}{6}$, either by making the necessary arrangements of the objects of the group or by thinking the arrangements. Later, when the pupils have developed sufficient confidence in their ability, they should be instructed and given practice in 'telling the story' about the group. To tell about six, for example, the pupil tells (and demonstrates or not, as may be required) "One from six is five," "One and five are six," "Four from six is two," etc. Or, he may 'write the story,'

as follows: $\frac{6}{5}$, $\frac{1}{6}$, $\frac{-4}{2}$, etc. In connection with a review

of six, what the pupils learned in connection with the exercises of comparison should be told and written: "Six is two more than four," "Six is three less than nine," etc.,

and $\frac{6}{2}$, $\frac{9}{3}$, etc.

In connection with the written review the point should again be emphasized that such a form as $\frac{-4}{6}$ means either 'more,' or 'less,' or 'take away.'

n. The Results of Studying the Arrangement of Groups. As

8							9						
8	8	8	8	8	8	8	9	9	9	9	9	9	9
-1	-2	-3	-4	-5	-6	-7	-1	-2	-3	-4	-5	-6	-7
7	6	5	4	3	2	1	8	7	6	5	4	3	2
1	2	3	4	5	6	7	1	2	3	4	5	6	7
7	6	5	4	3	2	1	8	7	6	5	4	3	2
8	8	8	8	8	8	8	9	9	9	9	9	9	9

Notice that no zero combinations are shown in these written expressions. The use of the zero in the number system will be discussed in a later chapter. The child has no use for the zero until he comes to the study of ten. Hence there is nothing gained at this time by introducing the zero.

a. Use of the Terms 'Subtraction' and 'Addition.' In the study of a group, the pupils begin with the whole group together. The first thing that can be done with it is to separate it in some way. In each case, a smaller group was first *taken away*. This is the process of subtraction. Now, when the original group which is being studied has been separated into two smaller ones, the two smaller groups were *put together*. This is the process of addition. In each case, an addition arrangement followed its related subtraction arrangement.

During the activities just described in Section 3 the teacher has emphasized and the pupils have given their attention to the process of *taking away* and the process of *putting together*. The pupils have actually performed the processes again and again, and they have repeatedly *thought things away* and *thought them together*. Such activities have led to a development of the ideas 'take away' and 'put together.' Now, when the ideas have been gained, the teacher can offer the usual names for the ideas; namely, 'subtraction' and 'addition.' Now, the names need no explaining, because the pupils have already gained the ideas for which the names stand.

p Directing Attention to Arrangement. It will be observed

that a given arrangement of objects ~~is~~ ^{is} ~~may be described as~~ ^{may be described as} ~~the~~ ^{the} ~~example, /// //~~ ^{example, /// //} — may be described as follows: ~~the~~ ^{the} ~~namely, "three and two are five."~~ ^{namely, "three and two are five."}

namely, "three and two are five," ~~the~~ ^{the} ~~and "two and three~~ ^{and "two and three}

are five," ~~the~~ ^{the} ~~3.~~ ^{3.} The later use of such a point of description

'combinations,' as they are commonly called, ~~often~~ ^{often} ~~enables~~ ^{enables} the teacher to require their learning as ~~an~~ ^{an} ~~aid~~ ^{aid} ~~of~~ ^{of} ~~experience.~~ ^{experience.} Their later use requires a mastery of each in isolation. A given situation will require the use of one combination without any reference to the other two. For this reason the teacher is sometimes in ~~consequence~~ ^{consequence} ~~embarrassed~~ ^{embarrassed} to present them together or separately.

The nature of the pupil's work at this point makes the question for the teacher. If the pupil is given his attention to a method of description as a fact of ~~experience~~ ^{experience} ~~he may be taking the idea of arrangement that is to be~~ ^{he may be taking the idea of arrangement that is to be} ~~posed to describe.~~ ^{posed to describe.} If he is given his attention to an arrangement of objects, the different methods of description will be thought of as belonging together. Their relations, ~~and~~ ^{and} ~~their~~ ^{their} ~~similarity~~ ^{similarity} — will have more significance than their difference. It is not greatly important whether one says "the pencil and paper are on the table," or "the paper and pencil are on the table." The situations being described ~~are~~ ^{are} ~~the~~ ^{the} ~~same.~~ ^{same.} "Three and two are five," ~~the~~ ^{the} ~~and "two and three~~ ^{and "two and three}

are five," ~~the~~ ^{the} ~~3.~~ ^{3.} are different only in form of expression. They

may be used to describe a single arrangement.

9. *Thinking, not Memorizing, should be Encouraged.* It can be repeated that the teacher should guard against the learning of the language of arrangement as mere facts of memory. If a pupil has difficulty in thinking and is given

instead of merely suggesting the right words or showing him the correct written expressions, help him to work out his difficulty for himself. If thinking the arrangements without resort to the objects is too difficult, let him take the group of objects and actually arrange them in various ways, describing aloud and writing each arrangement. Next, let him attend to each arrangement and describe each without actually making the arrangement. The direction of effort should be to the end that the pupil will *think the arrangement*, not merely remember something to say and something to write.

r. Summary. The purpose of the activities described in this Section (3) is to lead children to develop and enlarge and clarify their ideas of the numbers to nine. Since the idea of number is the idea of the group, the children (1) have been taught how to study a group, and (2) have been directed into a more or less independent study of groups. Since the systematic study of a group is more effective than haphazard, incidental study, they have been directed in methods of studying groups systematically.

Through a development of their number ideas to nine, the children have received preparation for the later activities of using these nine ideas in combination. And they have not merely received preparation; they have actually learned to carry through, and to organize in proper relations, nearly half of the total of addition and subtraction combinations. Thus, their learning has demonstrated the fact that a step in arithmetic not merely prepares for the next step, but is in reality a good part of it.

4. Studying Groups by Emphasizing Equal Groups

By means of the methods of studying groups described in the two sections immediately preceding, the pupils enlarge and greatly clarify the ideas of the numbers, one to nine, that they had previously developed through counting, and in this process they learn many number facts in their relations and in connection with the nine ideas. In their activities

opportunity also is afforded them to have, when such different number facts that serve the double purpose of reviewing the nine number ideas and of contributing to the development of the idea of groups. The number facts reviewed are the few simple facts of multiplication, such as two and two, three and three, etc., which contribute to the ideas, four, six, eight, and nine.

a. *Avoid Confusing Multiplication with Addition.* It is probably better for the teacher in commencing the work of arranging a group of eight than first to show four and four, six, four and four, etc., merely to call attention to the fact that four and four are equal groups than to attempt in the beginning to bring out the fact that in eight there are two fours in connection with the previous facts of addition. Mathematically there is no substantial difference between "two fours are eight" and "four and four are eight," because both facts contribute to the product eight of eight. Practically, however, there is a sharp distinction between the two facts, and the distinction is greater as the children learn to learn as the similarity. The common error of the children makes clear the point that "two fours are eight" and "two and three are five" must have the same or different and distinct arrangements before they can be brought together. It is the purpose of the present work to emphasize the point that the related facts "two fours are eight" and "four and four are eight" must not be mixed.

When one adds four and four he must not give attention to the fact that the two groups are equal. He relates the process to those of adding two and two, three and three, and so on, in which the groups added are not of the same size. If when the children are turned to the fact that in four and four the two groups are equal, he thinks of the combination as one of addition. He is compelled to view it as one of the very same process as addition. Indeed, the adding of groups of equal size is such an unconscious circumstance as to demand no special recognition at all.

On the other hand, when one multiplies he does so because he is impressed with the fact that he is dealing with groups of exactly equal size. The equal size of the groups is a fact of major importance in determining to resort to the process of multiplying. Moreover, the exact number of the groups is a matter of special concern in multiplication, whereas in addition one pays little attention to the number of the groups to be added and is not assisted in the process by the little attention he does pay.

b. *Addition and Multiplication Are Indirectly Related.* The two processes bear to each other an indirect relationship. The fact that they both are useful in contributing to certain number ideas, not the fact that they are similar processes, is

The Idea Eight the reason why they should be put together as related processes. The accompanying figure may serve to illustrate the point. Processes (1), (2), (3), and (4) all contribute to the enlarging and enriching of the idea eight.

$$\begin{array}{cccc} 5 \} \leftrightarrow \left\{ \begin{array}{c} 4 \\ 4 \\ 8 \end{array} \right. & & 4 \} \leftrightarrow \left\{ \begin{array}{c} 2 \\ 2 \\ 8 \end{array} \right. & & 2 \} \leftrightarrow \left\{ \begin{array}{c} 4 \\ 4 \\ 8 \end{array} \right. \\ (1) & (2) & (3) & (4) \end{array}$$

They all are held together by the contributions they make to a common idea. Processes 1 and 2 are directly related because they are similar processes; likewise Processes 3 and 4 have a similar relation. Processes 2 and Process 3 are similar only when and to the extent that one gives attention to the number of groups in (2) and to the fact that the groups are exactly the same size. Processes 2 and 3 belong together because they contribute to a common idea, not because they represent similar ideas of grouping.

c. *Teach Separately, then Relate.* There is no intention in the present discussion to suggest that all the activities of the preceding section (3) should be completed before those of this section (4) are begun. The arrangements "two threes are six" and "three twos are six" might just as well be learned before "four and three are seven," "two and five are seven," etc., as to be delayed until all of the arrangements of subtraction and addition that contribute to the

ideas one to nine are learned. The teacher should not be so much as well to delay the arrangements and multiplications as possible until they are finally made contributory to the multiplication process one to nine. As frequently pointed out in the preceding section, it is the number ideas, not mechanical facts and mechanical processes, to which pupils should give their attention from the beginning. The various forms of numbers, their relations should be learned, and in particular, they should be able to use as contributors to the number ideas. The teacher should be strained, not apparent, in his efforts to make the arrangements should be approved in. The pupils must learn to distinguish between them.

Whether the teacher introduces the multiplication of ideas at the outset, presents them later, or introduces them at the end of the section or keeps them entirely separate are all matters of great importance. The important principles to be kept in mind are: (1) whether the multiplication is of ideas, or of numbers, or of both, and (2) whether the multiplication is of ideas, or of numbers, or of both. The teacher should keep in mind that the multiplication of ideas is of ideas, and the multiplication of numbers is of numbers, and the multiplication of both is of both. The teacher should also keep in mind that the multiplication of ideas is of ideas, and the multiplication of numbers is of numbers, and the multiplication of both is of both. The teacher should also keep in mind that the multiplication of ideas is of ideas, and the multiplication of numbers is of numbers, and the multiplication of both is of both.

d. What to Emphasize. In introducing the multiplication of ideas, the teacher should develop the arrangements and multiplications of ideas, and the multiplication of numbers. The teacher should also keep in mind that the multiplication of ideas is of ideas, and the multiplication of numbers is of numbers, and the multiplication of both is of both. The teacher should also keep in mind that the multiplication of ideas is of ideas, and the multiplication of numbers is of numbers, and the multiplication of both is of both.

It may be that the pupils will have difficulty in distinguishing between multiplication and addition, and the teacher should keep in mind that the multiplication of ideas is of ideas, and the multiplication of numbers is of numbers, and the multiplication of both is of both. The teacher should also keep in mind that the multiplication of ideas is of ideas, and the multiplication of numbers is of numbers, and the multiplication of both is of both. The teacher should also keep in mind that the multiplication of ideas is of ideas, and the multiplication of numbers is of numbers, and the multiplication of both is of both.

at this point in the course that a beginning be made in developing the understanding of the pupils.

Frequently the attempt is made to lead pupils to derive the idea of multiplication from the arrangements of addition; in other words, to teach multiplication as a special form of 'short-cut addition.' From the arrangement "three and three are six," the idea is suggested, or attempted, that "two threes are six," and so on. Such attempt serves to emphasize the similarity between the two processes, but fails to emphasize sufficiently the distinction.

Beginners in arithmetic seem to be able to note with little difficulty the similarity between addition and multiplication.⁶ Questions relating to the two processes are asked in very similar terms, and both processes are 'put together' processes. Their chief difficulty is in distinguishing between the processes, and in avoiding being confused by the apparent and real similarities.

Another criticism of the method of presenting multiplication as 'short-cut addition' is that it unduly delays most of the multiplications that the pupils can otherwise readily learn. For example, the arrangement "three threes are nine" has to be postponed until the pupil learns to add a column of threes, and "nine sixes are fifty-four" has to be postponed until the pupil develops an ability in column addition that an understanding of the number system does not require. Many times pupils are found in the latter half of their fourth year with no more introduction to some of the simpler multiplications than purely memoriter exercises are able to give.

The contrast between the Roman and the Hindu-Arabic system of notation suggests another criticism. The Romans, 2000 years ago, when they wished to multiply, had to add. The system of numerals they had was a system of additions.

⁶ C. H. Judd. *Psychological Analysis of the Fundamentals of Arithmetic* (Department of Education, The University of Chicago, Chicago, 1927), p. 88.

The Hindu-Arabic system, which we have is a system of multiplications, as we have seen is a very complicated system. It seems very strange, to say the least, that teachers should have no better method of teaching multiplication than the method of multiplication that was in general use twenty years ago.

c. *The Study of Groups from the Point of View of the Pupil.*
The necessity for further discussion and improvement may be escaped if we turn our attention from a consideration of the problem of the teacher to a consideration of the work of the pupil. In the preceding section the question of whether addition comes first or subtraction comes first was answered entirely by confining our discussion chiefly to the learning processes of the pupil. The larger number discussion was the study of the group by the pupil. When he has a whole group, and undertakes to study the whole group the first thing he can do with it is take it apart or to consider groups. This is subtraction. Now, when he has taken apart the group being studied, he can next put the parts together. This is addition.

Perhaps, if we keep our attention upon the things the pupil has before him to study, the whole question of how to introduce the idea of multiplication multiplying numbers leads to the number of groups and the equality of groups will solve itself. Let us bear in mind that the pupil, as well as doing the so-called "combination" when he is engaged in the activities that have been described so far, is also proceeding. He is studying groups as objects in his mind, and as such, some of the groups. As the activities to be described in the present section, the pupil must continue to study groups. The study of the group, and the group as the center of attention will determine what the pupil learns first and the manner of his learning.

Let us review, for our own benefit, the suggestions previously cited that, when the pupil is engaged in the study of a group, the first thing he can do with the group is take it

apart into smaller groups, and the consequent fact that the putting together of the smaller groups must, of necessity, follow after the taking apart.

f. The Sequence of Taking Apart and Putting Together. Let us suggest that emphasis upon equal groups follow the activities that have been described in the preceding section. Let us suppose that the children are reviewing what they have found out about a group of eight, for example. Each child has eight objects, and each in turn is answering a question by the teacher about eight:

"Five from eight is how many?"
$$\begin{array}{r} 8 \\ -5 \\ \hline \end{array}$$

"Six and two are how many?"
$$\begin{array}{r} 6 \\ 2, \text{ etc.,} \end{array}$$

or each child is telling or writing 'the whole story' of eight.

When such review exercises have been completed, the teacher may introduce the new kind of study somewhat as follows:

The teacher takes a group of eight objects — *////////* — and makes the suggestion:

"Let us study something new about eight."

"How many pencils have I?" Eight.

"I would like to find out how many twos there are in eight. This is the way I can find out."

The teacher now separates the eight objects into groups of two.

"I want to find how many twos there are in eight, so I count out one two, and place them together here; I count out one two, and place them here; I count out one two, and place them here; and I count out one two, and they are here."

// // // //

"How many are here?" (pointing to the first two)

"How many are here?" (pointing to the second two), etc.

By question and suggestion the point is emphasized that each group is equal.

"Now I have my eight separated (*divided* is just as familiar

just discovered in answering the 'division' question, or by putting together or thinking together the four groups of two to make eight. The new question — the 'multiplication' one — serves both as a review, and as a new view, of the arrangement being studied.

Give both kinds of questions, such as "How many twos in eight?" and "Four twos are how many?" Since both require the same answer, each strengthens the other.

g. Writing the Question and Answer. The pupils may be shown the written form of the division question, and how to write the answer when they have found it, or the written form may be delayed until later. There is a slight advantage in showing the form now, since it will be of service in later reviews, and the written form, being different from ones already learned, may help to fix distinctions.

After the pupils have learned to determine for themselves answers to such questions as, "How many twos in eight?" they may be shown how the question is written, thus:

$$2\overline{)8}$$

Let it be impressed that the written form asks, "How many twos in eight?" Do not call it "division," "divide," or describe it in any other way. Now, when they know what the question asks, show them where and how to write the answer.

$$\begin{array}{r} 4 \\ 2\overline{)8} \\ \underline{8} \end{array}$$

As the 4 is written above, and another 8 below and the line is drawn, give the answer, "Four twos are eight."

The special written form of multiplication, $\begin{array}{r} 2 \\ 4 \\ \underline{8} \end{array}$, ought not to be introduced until later. It would be of service in connection with but few arrangements, and would probably be confused with the written form for addition. Since the writ-

in form of division, 2) 3, for example and in the operations needed by the pupils in their present study in geometry, the written form of multiplication is not needed.

Let it be repeated that, when the written form is presented, it be presented as a way of solving a problem. 3) 4. "How many tens in eight?" The answer is 2 and the pupils can work out and give it as given and it will need no further explanation.

4. Counting by Tens, Threes, Fours, and Fives. An experiment, though somewhat incidental, means of grouping information both to equal groups and to the idea of groups in the engaging activities of counting by tens and three and four. Pupils engage in these activities usually with one rather than two groups, that of intellectual play. Just as the teacher is interested in employing himself in counting, multiplying, the business of his class, the pupils in the class, the teacher in the class so the pupil who has learned to count by ones and tens learns a new way to count by tens and three and four groups, suggested in these new methods. A little encouragement from the teacher is all that is needed either to maintain the activity or to help them along. Encourage the pupils themselves to do it. It is a very good way to learn to count by ones and tens.

A good example of the pupils who quickly extend these activities to the counting by threes to thirty, and by fours to forty. Pupils who are usually engaged in with ones, twos, and threes do so much to call the attention of the pupils to groups which in the quality of the groups as any other (the teacher says so).

A word of caution may well be given at this point. It is important to be reminded that the pupils who are doing the number count in counting by ones for example, may be doing no more than organizing numbers. It is not as if they may not, recognize the whole idea (counting by ones) as a whole twelve and that two more than threes is the same. In Section I, it was pointed out that counting by ones is not the same as that gives meanings to numbers and numbers.

at once both the saying of names and the discrimination of objects. Similarly, at this point, it needs to be made clear that counting by twos, for example, is not only saying the names, two, four, six, etc., but also distinguishing objects in groups of two and applying the names successively to the discriminated groups. Therefore, do not ask your pupils to count by twos when you suggest no objects to be counted. Let them count books, crayons, seats, pupils, etc., by twos until the ideas of relation between six and eight, twelve and fourteen, etc., are entirely clear. Afterward, it is permissible to let them merely say the number names as a means of fixing their serial order. Bear in mind that it is ideas and meanings with their related names that are to be gained, not the number names apart from the ideas and meanings they are intended to express.

IV. SUMMARY

The activities which have been described in this chapter are the activities of studying systematically the groups of objects to nine. By means of such systematic study of groups, the pupils have built up and clarified and made familiar the number ideas to nine. They may now be called upon to learn the procedures of *using* these ideas in combination with each other. When they are introduced to the activities that follow, they will be able to use the number ideas, which now have become familiar, with confidence and understanding.

In connection with their study of groups to nine the pupils have been introduced to 36 combinations of addition, 36 of subtraction, 6 of multiplication, and 6 of division; and they have learned, with the exception of the 6 multiplication ones, the meaning and use of their corresponding written expressions. Thus, in the process of preparation for the study of combinations, the pupils have actually learned nearly half the addition and subtraction ones.

CHAPTER XI

PRACTICE FOR MATHS

Annotations

1. The 'problem' in arithmetic is a practical proposition. It should serve, not as an actual problem, as a 'mathematical' problem, but as a practical proposition.
2. The 'problem' in arithmetic is a practical proposition. It should serve, not as an actual problem, as a 'mathematical' problem, but as a practical proposition.
3. Later, when such ideas have been grasped, the student should be able to recognize them in familiar situations.
4. 'Problem-solving' in arithmetic is, first, the recognition (a) of general ideas in familiar situations and (b) of new situations with which the student is not familiar. Thus the idea that 'problem-solving' is distinguished.
5. In the selection of situations for practice the student should be able to provide the student with (a) a series of problems of increasing difficulty and (b) a series of problems of increasing difficulty.
6. The purpose of writing in arithmetic is to provide the student with a series of problems of increasing difficulty and (b) a series of problems of increasing difficulty.
7. Drill should be distinguished from practice. It should provide more than practice in the use of the techniques of arithmetic. It should provide practice in arithmetic.
8. Drill should be distinguished from practice. It should provide more than practice in the use of the techniques of arithmetic. It should provide practice in arithmetic.
9. The grade of drill is determined by the student's ability. It should provide more than practice in the use of the techniques of arithmetic. It should provide practice in arithmetic.
10. Throughout the arithmetic, the student should be able to provide the student with a series of problems of increasing difficulty and (b) a series of problems of increasing difficulty.

I. ATTENTION TO ARRANGEMENT

In the activities described in the preceding chapter, the pupils give their attention in studying groups to the arrangement of objects (1) when the teacher makes the arrangements, (2) when they themselves make the arrangements, and (3) when they think the arrangements. From step to step the pupils study the same things, but each step of the study is on a higher level, both of difficulty and of thinking required, than the one preceding. Now, having carried their study of arrangement to the level of thinking the arrangement, when the objects are present, the pupils should be led to take the steps of study that require the thinking of arrangement when the objects are not present. These we have designated as the steps of giving attention to the arrangement of objects (4) when the objects are present only in imagination, and (5) when no objects at all are present.

II. 'PROBLEM-SOLVING' AND 'DRILL'

The designations given to Steps 4 and 5 are not sufficient to enable one to distinguish the steps. It will be necessary to describe each in some detail. If one, however, uses the terms commonly employed to designate Steps 4 and 5; namely, 'problems' and 'combinations,' respectively, or 'problem-solving' and 'drill,' respectively, he has no difficulty in making the intended distinctions clear.

Although the common names for Steps 4 and 5 appear to set them off with sharp distinctions, the names in reality serve to emphasize distinctions that are more apparent than real. Steps 4 and 5 embody a continuation of the same kind of study of arrangement that was begun with Steps 1, 2, and 3. Throughout the first three steps the pupils studied the same thing; namely, arrangement; in the steps that succeed, they are to continue to study the same thing; namely, arrangement. Steps 1, 2, and 3 were exercises having a common purpose; Steps 4 and 5 are exercises having the

one common purpose, but different in form of the exercises introduced in the eye and that cause the change in the eye.

The terms 'problem-solving' and 'problem-solving' are used so long, and their use is so widespread that they are almost identical in meaning. The terms are used to avoid confusion.

The terms 'problem-solving' and 'problem-solving' are used to indicate that both 'problem-solving' and 'problem-solving' are used for practice, and that both have a common purpose, namely, that of providing students with a series of exercises in arrangement on levels higher than the usual exercises at the beginning of study.

III. Problem-Solving Exercises

1. The Purpose of Problem-Solving

Problems are often called 'examples', 'illustrations', or 'examples', and are used to illustrate a concept. Often problems are called 'examples' or 'illustrations' as the center around which the exercises are arranged. The purpose of the problem is to provide the student with a series of exercises in order of the stage. It is intended to provide the student with a series of exercises in order of the stage. It is intended to provide the student with a series of exercises in order of the stage. It is intended to provide the student with a series of exercises in order of the stage.

In the preceding chapter the purpose of the exercises was to get the children to develop their ideas of the exercises, and to use, through the study of the exercises, to use. The teacher has presented subjects in a series of exercises and has taken special pains to make the exercises of the pupils the exercises in general. The purpose of the exercises is to provide the pupils with a series of exercises in order of the stage.

¹ Note similar discussion in *The Elementary School*, *Mathematics*, and *Mathematics*, Chap. IV.

arrangements of objects. Moreover, the pupils have at times, through the use of the language of number, thought the arrangement of objects without actually disturbing them. It is a further step in the pupils' development to think the arrangement of objects when the objects are present only in their imaginations.

In the problem "John caught 5 fish and gave Henry 2. How many did John have left?" the pupils are given a situation not actually present to the senses. The interest of the pupils is in the total situation, to be sure; they have, however, in the number relations -- that is, in the rearrangement of the objects -- a very special interest. Heretofore, the pupils have had before them five objects actually present and have laid aside two and observed three objects remaining. They have described the arrangement orally thus:

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"Two from five is three," and in writing thus: $\begin{array}{r} -2. \\ 5 \\ \hline 3 \end{array}$. In

the problem relating to John's fish, the pupils are compelled to think the rearrangement of the five fish. They can give little or no attention to the fish as individual objects. If the numbers were larger, they could not visualize the objects except in a vague, general, and indefinite outline of the whole group. Without giving attention to the fish, the pupils give attention to their division into two groups, employing the language of arrangement, which they have developed in connection with the actual arrangement of objects, to describe the arrangement now present only as a matter of thought. Because they have been guided through language to attend to the grouping of objects actually before them, they have learned to think in exact terms of a similar situation, not actually before them.

Moreover, dealing with a situation not actually present enlarges the pupils' ideas of arrangement. In the activities with objects, "two objects from five objects," for example, has been demonstrated repeatedly. The pupils have taken

given orally, "how many twos are in six?" or in writing, $2\overline{)6}$, and the pupils first observe and then perform the activity of resolving a group of six into groups of two. They discover the answer, and learn to give it orally, "three twos

3

are six," or in writing, $2\overline{)6}$. Thus the pupils learn the sig-

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nificance of the 'division' question, and how to find the 'multiplication' answer. In this way the ideas of division and multiplication begin to develop a long time before the common names for the ideas are given, and when the names are finally introduced, they need no explanation.

The importance of developing meanings in connection with processes being learned, and of understanding as a prelude to the habituation of processes, cannot be too strongly emphasized. It will be implied throughout our discussions as a fundamental principle to guide the organization of the course in arithmetic that, once a pupil grasps a meaning or develops an idea, he may be expected to gain an insight into number relations quite independently of the class activities. The meaning, or the idea, is an inner drive that both influences the pupil to think for himself and gives him power to do so. It works retroactively upon things already learned, giving them a new significance and a new interest; and it leads the mind to seek new truths that may serve to give it completeness. Meanings and ideas furnish a motive for learning, that, because it is logical and is intrinsically a part of experience already gained, transcends in the force of its influence any practical motive that is intrinsically a part of the world of experience not yet entered upon by the individual.

A fundamental fact about general ideas is that they develop slowly. The teacher does not give them. Their names do not necessarily carry them. They develop out of an ordered experience. They become familiar through constant contact. The teacher who has guided pupils through such

4. The Purpose of Problem-Solving Distinguished

Farquising discussions have indicated that the problem-practice exercise is useful for purposes of illustration. It will be necessary, therefore, for the teacher at all times to be very clear in his own mind as to just what the exercise is intended to illustrate. The teacher must have in mind the question: Is the exercise intended to serve as an illustration of a general idea of arrangement, such as addition (or, later on, percentage) or is the exercise intended to make clear by additional illustration a situation, such as square measure, interest, insurance, and the like? And the teacher's answer must be in accord with the state of the pupil's preparation and needs at the point where the illustrative problem-practice exercises in question are introduced.

Following such activities as are described in the preceding chapter, the pupil is prepared to profit by a number of suitable illustrations of the general ideas of addition and subtraction and perhaps by a few of the ideas of multiplication and division. The pupil is on no better than speaking terms with the ideas. He has met them only in the actual classroom situations incident to the study of groups. He needs now to have the ideas presented to him by a variety of situations that he can imagine in quick succession as a result of the oral descriptions of the teacher. He needs to enlarge his acquaintance with the general ideas by meeting them and recognizing them under many different circumstances. Since the general ideas are relatively unfamiliar and the purpose is to make them familiar, the situations that are described as a means of presenting the ideas for the pupil to recognize must be familiar situations. Since the general idea of addition, let us say, is a constituent of the total situation presented by the problem exercise, the situation, if a familiar one, will make the idea of addition stand out with sufficient clearness to be recognized. And since the situation is slightly different from any heretofore considered by the

pupil, the idea of addition is presented from a slightly different angle. We may summarize by saying that the purpose of problem-solving in the earlier stages of the learning of arithmetic is to develop an understanding of the meaning and use of the fundamental ideas of addition, subtraction, multiplication, and division.

In the later stages of the learning of arithmetic, it is as important for the pupil to develop an understanding of a number of the practical situations of life that involve the use of number relations, such as weights and measures, savings, budgets, interest, and the like. These situations are not the familiar ones of his everyday experience, just such as the circumstances of life as a modern society require him to learn, it will be necessary for him to make special studies of them when the time comes. In all these situations, the general ideas of number relations appear as constituents. Now, if the pupil can approach a study of these situations with an understanding of the meaning and use of the general ideas, he may encounter less serious material handicap or hindrance, and with a good many suggestions as to the nature of the situations. The general ideas of mathematics clear, will help to clarify acquaintance with the new situations. In the earlier stages, general ideas are developed through consideration of familiar situations. In the later stages — that is, when and if the general ideas have been developed — new situations may become the subjects of special study. The twofold purpose of the so-called 'problem-solving' activity in arithmetic must not be forgotten.

5. The Familiar Situation

Attention to the distinction between the earlier and later stages in the learning of arithmetic both indicates the importance of developing general ideas during the earlier period and impresses the importance of the use of familiar situations in the exercises of the earlier period. Consideration of the folly of attempting to teach new and unfamiliar ideas

tions before the general ideas of the processes that appear in them have been learned suggests the importance of the general ideas. Consideration of the need for general ideas as a prelude to the study of unfamiliar situations suggests the need for familiar situations at the outset as a means of making the ideas clear.

Moreover, the need for variety of familiar situations is evident. To enlarge his acquaintance with the general ideas, it is necessary for the pupil to meet them as different situations, to come upon them from different views and in different garbs. Continuity of the one suggests variations of the other. Since it is the general ideas that are the subjects of study and that should be at the center of attention, the situations that present them cannot well be continuous either in form or subject matter. If one is interested in the situations as ends in themselves, he will see to it that they themselves make a continued story. Such situations as 'the bean bag game,' 'a trip to the country,' 'the county fair,' and the like, will be the subjects of study and discussion. If, on the other hand, one is interested in the general ideas to be developed, he will select first one situation, then a different one, as illustrations of the ideas. He will see to it that the situation of a given problem is different from the one preceding and the one following.

Finally, no elaborate statement of the situation in a problem will be attempted. Two things are to be kept in mind. The first is that the situation should be a familiar one, in which case it does not need elaborate statement. The second is that the purpose of the exercise is to center the attention of the pupil upon an idea of arrangement, and not upon the possibilities of the situation. One may, if he wishes, add to the general enthusiasm in the situation by giving it an elaborate and interesting statement. One may, in such a problem as "John's fish," for example, describe at length John's fishing trip and the exciting adventures John experienced in catching the five fish, and he may stimulate

a lively discussion about fishing in general. Perhaps a generalizability offers its temptations to every teacher. It is so easy to digress from the work at hand, and so interesting to arouse the interest and enthusiasm of the pupils that one can hardly resist leaving the general idea of water-sports. For example, for an animated discussion of an important thing on his vacation. But remember that the interest that may be aroused by such digressions is an outer, superficial interest, not intrinsic interest in what should be the central theme of the exercise. The present writer has elsewhere brought together data showing that the effort to arouse pupils in their problem-practice exercises by elaborate statements of the situation is largely misdirected and wasted.¹

6 Ready Recognition Is Required

The point is often made that, in the actual and real experiences of life, one is confronted by very much more of the whole of a given situation than is presented in the usual conventional statement of the problem-practice exercise such as our discussion has recommended and such as text books usually offer. A very pointed presentation of this view is offered in an article by H. G. Whist, from which the following paragraphs are taken:

Many of the problems that arise in life can be solved only by the application of quantitative methods, involving such quantity computation but also the consideration of important social and economic relations. For example, suppose that a sidewalk in a state of disrepair is to be replaced. The problem the owner of the property faces is, 'How can this be done most economically and efficiently?' The required verbal problem in arithmetic which presents this situation to the pupil might be stated as follows: 'Mr. Anderson plans to build a sidewalk 5 feet wide and 45 feet long. How much will this

¹ H. G. Whist, *The Relative Merits of Conventional and Imaginative Types of Problems in Arithmetic* (Bureau of Pedagogical Research, Teachers College, Columbia University, New York, 1929).

cost at \$2.25 a square yard? In this statement the pupil is given all of the essential facts needed to get the answer to the question asked. He must decide on the processes to use and then perform the necessary computations. Some of the important social and civic relations involved in the situation as it would develop in life are introduced in the statement of this typical verbal problem.

The questions that the property owner himself must face are much more far-reaching than those included in the above problem. Such considerations as the following arise in the typical situation: "Are there any legal restrictions on the type of sidewalk that may be laid, its width and materials of which it may be constructed? If so, why are these restrictions of this kind? Does the city build the sidewalk or must I secure a contractor myself? If the latter, how do I go about it to select a contractor? How do I secure bids on the work to be done? What information should I have concerning the reliability of the various firms which will guide me in selecting the firm to do the work? How can I be certain that the contract is correctly drawn? Are the bids too high? What investigations can I make which will enable me to find out if the bids are too high? In the absence of legal specifications how can I be certain that my sidewalk will be constructed according to sound engineering principles? After I have let the contract, need I check in any way on the extent to which the specifications in the contract are carried out? What steps will conclude the transaction? Does the city pay any portion of the cost of the sidewalk?" Similar questions are faced by the contractor in making out the bid so as to secure the work, in buying materials, laying the sidewalk, and supervising the work.¹

The situation of sidewalk building takes us far beyond any activity suitable for beginners in arithmetic, but the manner of its discussion is illustrative of the kind of problem the teacher of beginners must face. Is the typical verbal problem of the cost of a sidewalk 3 feet wide and 45 feet

¹ L. J. Brueckner. "The nature of problem-solving" (*Journal of the National Education Association*, 21: January, 1932), pp. 13-14.

in order that he may move on to real-life experiences on a level higher than those of childhood. We see the need for the development of the idea of a particular kind of number relation, and we provide for its abundant and varied illustration. We look beyond the pupil's present life to the time when he will be able to use his attention with certainty upon a particular type of number relationship, and we provide for a larger and more involved total situation; we note that the usual life experience does not require a concentration of attention upon a particular fact of number, but we provide as an elaborate statement of Frank's pennies might require; and so we provide for the pupil's training the type of illustration that demands a ready recognition of the general idea that it portrays. We remember that the exercises that are offered are intended to give training in the recognition of ideas; and so we select those that are varied for the sake of breadth and those that are tersely stated for the sake of readiness.

7. The Purpose of Writing

Frequently the efforts of the pupils to arrive at the answers to the questions raised in problem-practice exercises are guided in the direction of an excessive amount of activity with pencil and paper or at the board. Too often the pupils are required to engage in writing to such a degree that they quickly form the notion that problems are solved on paper or on the board. It is the too common occurrence for pupils to begin writing before they begin to think, and even to try first one operation, then another, without indulging in thought. It is important for us at this point, then, to get clear concerning the real purpose of the activity of writing in connection with the mental activity of finding the answer to a problem.

In the solution of a simple, one-step problem, the adult does not need to write. He solves it 'in his head.' In the solution of a problem of several steps, he needs to do some

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writing, but here again he writes it "in his head". He is able to take each step correctly, but he finds it necessary, in order to take several steps consecutively, to write the result of each step as a means of remembering it while he

is working.

He is now able to

do the following:

He can do

the following:

He can do

the following:

He can do

the following:

He can do

the following:

He can do

the following:

He can do

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He can do

the following:

In a problem that requires a two-place operation, such as "there are 22 boys and 23 girls in the class," the pupil

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will at the outset need to resort to writing, 23. He first adds 2 and 3, and sets down that partial result, which stands as a reminder of what he has done. No effort is then needed to remember that partial result, because it is before him in writing, and in its proper position; he is able to turn to the next requirement of the solution with a free mind. He now adds 2 and 2 and sets down that partial result in its proper position. The two partial results, set down in their proper positions, give him the answer he seeks.

In a problem that requires addition in the higher decades,

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such as 3, writing is not needed in the activity of determining the answer. The addition that is made ought to be a single process, like the adding of 5 and 3. Writing should not be permitted, except for the purpose of establishing the connection between the solution and the written record.

What has been said about the purpose of writing in the exercises with the simple one-step problems of the pupil's earliest activities applies with like emphasis to his later activities with problems of two and more steps. These later exercises are intended to give practice in recognizing the ideas of addition, subtraction, multiplication, and division when they appear as double and triple operations, etc. The real purpose of the exercises is to give practice in recognition of ideas. The pupil needs to write only enough of his computations to serve as a sufficient reminder of what he has done mentally at each step to permit him without hindrance to proceed to the consideration of succeeding steps.

8. How to Conduct the Practice

The purpose of the problem-practice exercise will bear repeating: it is to develop ideas by means of ready recognition of ideas in familiar situations. To this end, let the

practice for the most part be oral practice. Let the teacher or a pupil state the problem, and a pupil give the answer. When the pupils have had considerable oral practice, it will be time to show how the solution is recorded on paper as on the board. Only enough of the written answer should be written to show the connection between the averages in the problem of "Frank's journey."

and the way it is recorded. If the pupils have

done in a satisfactory way the work suggested in the preceding chapters, they will be able to

have a

when

When a problem is read aloud, it should be read at least. The adult who is confronted with a written problem reads it twice or more. He reads it the first time to grasp the import of the situation that is presented. The next, of course, require more than one reading. He reads it the second time (or finally) to recognize the numbers exactly. So, in presenting a problem, the teacher should read it to the class once in order for the pupils to grasp the meaning. (If one reading does not make the meaning clear to the pupils, read it again, and even a third time.) Permit no pupil to write a number during the first reading. Permit no writing until the pupils understand what is presented in the problem and what is required. When they understand, read the problem a second time (or finally) for them to get the numbers exactly.

Encourage a minimum of writing. If a pupil "works" a problem without the need for writing, he has secured all the practice the problem is intended to give. Commend him for his accomplishment. Let him make a note of his answer (write it) if the oral answer is not desired. It will be necessary, however, to show, for the pupil to grasp the quantities in the problems involving columns with three, two-place numbers, and the like.

Moreover, the pupils should be taught to avoid unnecessary labeling. A label in the written record of a solution is sometimes needed to serve as a reminder. When needed in the more involved problems of later arithmetic, it should be so used. In the simple, one-step problems of beginning arithmetic, there is no such need for labels. What the quantities stand for is comprehended without the labels. Moreover, the requirement of writing the labels serves to impress the mistaken notion that the solution of a problem comes through writing.

9. Summary

When the pupil has had sufficient practice in thinking the arrangement of objects to understand what he is about, he may proceed to practice in thinking arrangement when the objects are present only in imagination. He thus is enabled to carry his practice to a higher level. Moreover, he is provided practice in the recognition of certain general ideas of arrangement that appear in a variety of familiar situations to which he must give his attention.

Such practice in the recognition of general ideas in familiar situations is the so-called 'problem-solving' activity on the early levels of training. Later discussions will give further consideration to this phase of the problem-solving activity and also to the phase of the activity that is appropriate on the later levels of training.

IV. PRACTICING THE COMBINATIONS

1. Attention to Arrangement

In the activities of studying groups described in the preceding chapter, the pupils give their attention to the arrangement of objects (1) when the teacher makes the arrangements, (2) when they themselves make the arrangements, and (3) when they think the arrangements. In the activities, described in the preceding section of this chapter, the pupils carry their study of arrangement to a higher level.

They learn to give their attention to arrangement & place the objects to be arranged not merely and so mechanically. Now, in order to carry their thinking to a high degree, the pupils must have practice in giving their attention to arrangement (5) when no objects at all are present. At this last step the study of arrangement is entirely a matter of thought. By means of the language of arrangement when the pupils are in the last stage preceding, they practice arrangement and give them

Ex: 1. 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100

2. What Practice Should Mean

Practicing the combinations often means nothing. It often degenerates into mere practicing in writing the symbols and saying the appropriate words. When the pupil practices writing $\frac{1}{2}$, and saying "Five and three are eight,"

without giving any attention whatever to the idea of arrangement for which the expression stands. When he does this, he receives practice, to be sure, but it is merely practice in using written and oral expressions, and not practice in thinking. Such practice leads to time in perfect automatization of the process. Whether such practice has any value is a matter of serious doubt. About the fact that it produces harmful results, however, there can be no question. Pupils often learn to "add" — that is, to go through the motions that bring the right answer, when the meaning of adding has escaped them entirely. They know how to use adding machines, able to add as fast as the machine in doing the work, but unable to decide when and why to add.

The kind of practice that should be conducted is not the kind that will be designed to bring the pupil into competition with an adding machine. The pupil can never hope to beat the machine in doing the purely

mechanical things the machine is made to do. The kind of practice the pupil receives should lead him to do the things that are mental, not merely mechanical. For his own thinking, however, a certain degree of mechanical perfection will be necessary. For example, he will need to be accurate in his work, and he will need to work with reasonable speed. But these are secondary to the all-important outcome of being able to understand. Moreover, if the pupil understands what he is doing, and then practices what he understands in an understanding way, he will in time reach the needed requirements of accuracy and speed. If we are going to think of accuracy and speed as outcomes of practice, let us think of them from the beginning in their proper relations with understanding, and in their proper sequence with it; namely, (1) understanding, (2) accuracy, (3) speed.

3. Dangers to Be Avoided

In conducting practice with the combinations, the teacher is constantly tempted to expect of the beginner the same type of automatic response that the adult has learned to give. The fact that a child can learn to write the expres-

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sion, 4, for example, with speed, and to speak the sentence,

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"Five and four are nine," with no hesitation, constantly suggests to the teacher the notion that the child can make the same response as the adult. It is a case of a similarity of outward responses suggesting a similarity of inward responses.

Through long experience with groups and their arrangements the adult has reached the point where his expressions of the combinations are far removed from concrete experiences. He has built up so much meaning back of the expressions that the expressions have come to stand in place of their meanings. Not only have the expressions themselves become habituated, but the directions of one's thinking that

accompany them have also become independent. It is to give the experience is sufficient for the child. The child on the other hand, has not acquired the same experience with the same amount of time. The child has been able to remove them very far from the experience for which they stand. These concepts have just begun to develop in his mind, and they have not become habituated. So, if the child is required to give the written and oral expressions with too great speed and without some reference to the concrete experience that makes them meaningful, he is tempted to forget them completely and to habituate the experience merely as things to write and as things to say.

It will be necessary, therefore, for the teacher to make haste slowly. It is a fairly safe rule never to let the child write anything or say anything unless he first understands what he is doing. When a pupil is given a question, the teacher should see that he fully understands the question and thinks the answer before he attempts to give the answer. The teacher should constantly bear in mind that the child can memorize all the written and oral expressions perfectly without giving a thought to the arrangements in which they stand.

4. Rules to Follow

a. *Make haste slowly.* Do not strive for speed. As the pupil learns to think through an arrangement, teach him to think through it again and again in the question to gain speed in giving the correct response.

b. *Delay the pupil's response until he is sure of the response.* If the student of arithmetic learns to think before he writes, he will save an enormous amount of time usually required for erasures and for repeated trials.

c. *Develop confidence.* Let the pupil be sure of his right, then go ahead. Confidence, which grows with understanding

ing, is always better than mere skill. Let the activities of this section follow the four steps in the study of arrangement that have been described in preceding discussions, and the pupil will be prepared to determine his own answers in the practice of the fifth step. Let the pupil take what time he needs to think the arrangement when no objects are present, and he will be sure of his answers. He will not only carry on successful work at the moment, but will also develop an attitude that will be most helpful at succeeding stages of his work.

d. Do not expect adult performance of children. If, when

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the child is given such a question as $7-3$, he resorts to the actual arrangement of the objects that are before him, or takes the time to imagine the arrangement, or counts off his answer on his fingers, let him proceed to his answer in his own way. Understanding of his answer and confidence in it are the important results to be sought. If he seems slow, have patience. Ask him the same question once more presently, and ask it again and again. He may still be slow in determining his answer, but he will gain in speed as practice continues. As he improves slowly through the right kind of practice, he will learn to substitute better methods for his earlier immature ones. It is a mistake for another to try to do the substituting for him.

e. Insist upon attentive practice. Be sure the pupil gives his attention to the question that is asked, whether given orally or in writing. Let him take the time to understand what is wanted and to determine what is wanted. His response should be fundamentally a thought response, not merely a verbal or a symbolic one.

f. Do not let the practice period drag. Start with good attention, and stop before the interest lags. When a question is given, insist that each child *think* the answer for himself and be ready to give it, if called upon. Call upon different pupils for the answers.

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... a *single specimen* must *keep*
or groups of specimens
and have knowledge on them
are mentioned.

h. *People want to know* ... *the real situation* ... *repeated words* ...

3. *Explain the ...*

The *whole* ...
leading to ...
to make ...
given their ...
various ...
arrangements ...
those who ...
ment ...

In connection with ...
 'tell the whole story of ...'
 six. The 'whole story' ...
 as the following:

- | | | | | | | |
|------------------|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| (a) Comparison | 1 | 2 | 3 | 4 | 5 | 6 |
| (b) Take away | 1 | 2 | 3 | 4 | 5 | 6 |
| (c) Put together | 1 | 2 | 3 | 4 | 5 | 6 |
| (d) Equal groups | 1 | 2 | 3 | 4 | 5 | 6 |

In connection with ...
 the pupils may either ...

6. Practice Upon ...

Practice upon the ...
 ideas that hold them ...
 are pretty definitely fixed ...

practice, practice upon the combinations in chance order should gradually be introduced.

It will be necessary eventually for the pupils to be able to use any combination under any circumstance, without reference to others that may be related. For example, the pupil will need to add five and three and to do it immediately without reference to two and six, or four and four, etc. He will need, when he is ready for it, practice upon five and three in isolation; that is, without reference to two and six, etc. Exercises that provide for such drills in isolation should be conducted. Let each pupil give the answers orally, across the lines, sometimes forward, sometimes backward, up the columns, and down the columns; or let one pupil give the answers for one row or one column, and let another give the answers for the next, and so on. After sufficient oral practice, let the answers be given in writing. It is desirable to use printed, or mimeographed, sheets containing the exercises for the written practice.⁴

7. Practice in Thinking

Let it be repeated that the practice desired is practice in thinking, not practice in remembering. Train the pupils so that they will keep themselves from speaking or writing an answer until they have thought out the arrangement and are sure of their answer. Practice in thinking will gradually lead to the type of memorization that is useful, whereas practice in remembering may defeat its own ends, because the pupils may try to remember things to say and things to write. Such practice in remembering is always confusing. Practice in thinking the arrangements is slow work at first, but speed gradually comes. What is more to the point, the pupils finally automatize their *thinking*, rather than their words and symbols alone.

⁴ The author's *Practice Books for Arithmetic*, Grade II to Grade VIII, inclusive (D. C. Heath and Company, Boston, 1936), provide such materials, particularly in the books for the earlier grades.

8. Using Devices

For the sake of adding a little variety it is permissible for the teacher to make use of various mechanical drill devices, such as 'getting up and down', 'using the fish pond,' 'spring around the circle,' 'stepping out of the maze,' and others of like nature.

The value of such devices, as indicated, is that they provide a little variety, or give, as it were, a change. They add flavor to the drills, as it were. The reason to them is to be recommended. A few distracting suggestions in one's work are not to be recommended.

Such devices, however, must be used with great discretion. Just because they provide variety and excitement they may prove to be too diverting. The child may become enthusiastic about a new trick. He is apt to become so absorbed in it as to miss the purpose for which it is used. For example, the child is apt to become so absorbed in getting the largest possible number of answers 'from the fish pond,' or in the imagined difficulty of finding his way out of the 'maze,' that he will forget to the purpose of reciting the answers to the combinations. Instead of thinking the arrangement of objects that the question asks for, he may become absorbed in the investigation provided by the game. If the pupils have interest in the arrangements of objects, a little game may be added to the exercises; too much, however, may spoil the exercises. If the children need to be diverted from the arrangements of the exercises, it will probably be better to stop the exercises for a while and let them get more and more interested in playing a game that has no connection with the exercises than to mix game and exercises and have a poor showing of both.

9. Providing Variety

The kind of variety to be provided, when variety is needed, is variety of method that will cause the same responses and

thinking, not variety of method that will lead to different responses. The type of response to be secured is, as pointed out repeatedly, that of *thinking arrangement*.

Variety, to list a few suggestions, may be had by such means as these:

- (1) 'Telling the story.'
- (2) 'Writing the story.'
- (3) Giving oral answers to exercises.
- (4) Giving written answers to exercises.
- (5) Using the blackboard or printed or mimeographed exercises.
- (6) Using number cards. On the face of the card the question may be written and on the reverse the desired grouping may be indicated, thus:



The reverse sides of the cards may be used for occasional reference, as needed.

(7) Illustrating numbers by distances. Additions and subtractions may frequently be illustrated to good advantage by directions of movement through spaces. For the beginner, the stepladder illustration may be suggestive. Let a ladder be drawn, nine steps high, with each step numbered. Let the activity be that of an imagined painter moving up and down the ladder while he is painting the side of a house. From the ground he moves up three steps. Next, he 'goes up' four steps. "How many steps up the ladder then?" "Now, he comes down five steps. Where is he?" "Now, he goes up six steps. Where is he?" And so on.

Indicate, before a question is asked, on which step the painter is standing. Be sure that is clear. Cover the numbers of the steps, and ask the question. When the answer is given, uncover the numbers so the answer can be checked.

It is to be remembered, whenever such devices are used, that the simpler the device the more quickly the pupil will understand its use and the less likelihood there will be that he will be distracted from his practice in thinking.

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10 Summary

The purpose of this book is to provide a summary of the concepts and principles of the various branches of the physical sciences, and to show how they are interrelated. The book is intended for use as a text or a reference work for students of the physical sciences, and for those who are interested in the general principles of the physical sciences.

CHAPTER XII

THE IDEA OF TEN. GROUPING BY TENS

Argument

1. Following the study of groups to ten, pupils must learn to think groups together and apart in a new way, namely, in relation to the idea of ten. Accordingly, this idea must first be developed.
2. The decimal system of notation requires the new and different method of thinking.
3. The place value of the numerals and the significance of zero as a place holder should be given consideration.
4. The so-called 'zero combinations' have no place in the elementary arithmetic of the Hindu-Arabic system.
5. Ten as a group should be studied in the same way that the groups preceding it have been studied (see Chapter X).
6. The use of ten in the writing of the 'teens and other two-place numbers should be learned. The significance of the ten's position should be called to attention.
7. The significance of the nine numerals in ten's position should be demonstrated objectively by groups of ten.

I. THE RESULTS OF THE PREVIOUS STUDY BY THE PUPILS

The activities in which the pupils have been engaging up to this point have been described as *thoughtful* activities — activities of giving attention in a thoughtful way to groups and to arrangements. To the extent that their activities have been thoughtful, the pupils have gained certain well-defined results. These we may enumerate briefly as follows:

1. Through the systematic study of groups to nine — by counting, by comparison, and by taking apart and putting together — the pupils have developed and clarified their number ideas to nine. By now, these ideas have become

definite. Each is an idea of organization as well as in its relations to the others, and it is so well known that it can be used with independence.

2 In the systematic study of each group has been arranged in various ways and put together -- and the arrangements studied, thought, described, and changed in passing. These arrangements were learned in their relations, remembered as their relations, and finally, returned and used as principles. Moreover, by means of problem-solving exercises the pupils have had practice in recognizing the value of combinations that go by the name addition and subtraction, as these appeared as integral parts of various situations.

By means of their earlier counting exercises and through comparisons, the pupils have grasped the idea of ten. The group of ten has been considered by counting it and by comparing it with smaller groups. Moreover, to the degree that the counting exercises have gone beyond ten, the pupils have gained other ideas. They have had some practice in counting to twenty, and they have had some practice in writing the signs or figures for the number ideas, ten to twenty.

II. The Next Step

In an earlier discussion the mistake was pointed out of trying to take children too rapidly and abstractly from the activities of counting to the activities of naming numbers alone in combinations. It was pointed out that the pupils were to develop and clarify their ideas in such a way that they can be expected to use them in combinations with understanding and independence. Accordingly, the systematic study of groups to nine was undertaken, which has had much use in development of the ideas, but also in some practice in using them in combination.

Now that the pupils have developed their ideas in names

and have learned nearly half of the addition and subtraction combinations of these nine ideas, it would seem that the next step is the one of learning the rest of the combinations of the nine number ideas.

Learning the rest of the combinations would, indeed, be the next step, were it not for the fact that the pupils must learn to deal with groups and with arrangements that exceed nine in a way that is quite different from the way they have been dealing with the groups and arrangements that do not exceed nine. From now on, the pupils must learn to group and to arrange objects, or to think such groups and arrangements, in a new way — new to them, at least. So, before they are led to undertake the learning of the rest of the combinations of the number ideas they have been learning, they need first of all to become acquainted with this new way of dealing with groups and arrangements of objects.

III. THE WRITTEN LANGUAGE OF NUMBER

The practice of expressing the number ideas in writing — that is, by signs, symbols, or figures — serves two purposes. In the first place, the writing serves as a record of thinking after it has been carried on. When one has counted a group of fifty objects, for example, or has come by some other means to an idea of this group, he can write down as a record the result of his thinking, thus, 50, or L, or fifty. As a record, each is as good as the other. Instead of having to carry the result in memory, he can turn to the record — 50, or L, or fifty — and note what he has once done.

Secondly, the writing of a number idea may serve, not only as a reminder of *what* one has thought out, but also as a reminder of *how* he thought it out. It may serve to record not only the *results* of thinking, but also the *manner* of thinking. Recording *results* aids the memory; recording *manner* of thinking aids later thinking, makes later thinking easier. Let us illustrate.

The sign, L, records fifty, but it does not show how one arrived at the idea. The sign, XL, records fifty and shows how one has built up his idea to get fifty. It shows four tens. One may hold in mind that L stands for five times X, but one does not need to remember that $50 = 5 \times 10$ or 10×5 . The relation between 50 and 10 is shown in the way 50 is written. The Roman sign, C, stands for one hundred, but it does not show ten tens. The Arabic sign, 100, both stands for one hundred, and shows ten tens.

In the Arabic system of writing numbers, there are nine signs, 1, 2, 3, 4, 5, 6, 7, 8, 9. Each stands for a separate group. Although the groups are related in various ways, the signs do not show the relations. One has to carry these relations in his mind. In the writing of the arabic group beyond nine, the same nine symbols are used in various ways, not to show different unrelated groups but to show tens and groups of ten. Everything beyond nine is written in so many tens. Instead of having different symbols for ten, fifty, one hundred, and five hundred, and so on, one uses the same symbols to write these ideas as he has learned to use in writing the ideas to ten. The symbols still are written in the same way and may be handled in the same way. They record the different ideas by the positions in which they are written.

The symbol 1 written in ten's place means 1 ten 10

The symbol 5 written in ten's place means 5 tens 50

The symbol 1 written in hundred's place means 1 hundred 100

The symbol 5 written in hundred's place means 5 hundred 500

To write eleven, one writes two 1's, 11. The signs, 11, may look like two 1's but they do not stand for two 1's. Position must be carefully kept. The signs, 11, show 1 ten and 1. The signs, 55, show 5 tens and 5, 555 show 5 hundreds and 5, and so on. Such considerations as these are of special significance in the training of the pupil because they emphasize the tools of thought that he should learn.

IV. THE USE OF A PLACE HOLDER

To write five hundred thirty-seven, one writes 5 in hundred's place, 3 in ten's place, and 7 in unit's place — 537. To write five hundred, one writes 5 in hundred's place — 500. Since one has nothing else to write, he must use some means of putting the 5 in hundred's place. The zeros, 0 0, hold ten's and unit's places. To write five hundred seven, one writes 5 in hundred's place and 7 in unit's place. The symbols written are 5 and 7. Suppose he writes it thus, 5 7. So written, it is confusing. One needs a sign to put 5 in its proper place. The zero does it, thus — 507. In writing a large number, 2463, for example, in which every place is filled by the sign for a number, one does not need a place holder, because each number sign holds its own place. But in writing a large number, 2060, for example, in which every place is not filled by the sign of a number, one needs something to fill up the places and hold them. The zero serves as a place holder. That is the use it has in our number system. That is the use the pupils need to learn.

V. THE SO-CALLED 'ZERO COMBINATIONS'

If pupils are taught the real meaning and use of the zero, they need not try to learn the various meaningless and misleading expressions that usually go by the name 'zero combinations.' They will have no use for such expressions, either in connection with their day by day experiences with groups or in their later study and use of the number system.

The pupil has no need for a sign for 'nothing.' If he wishes to write nothing, he writes nothing; that is, he does not write anything. If, in dealing with groups, he observes no group, he needs to pay the fact no further attention, and therefore, he does not need to say anything, or to write something, about it. To illustrate, if there are five marbles in one hand, and none in the other, the pupil knows at once without further consideration that the total is five. He knows it perfectly well before anyone tells him that "five

and nothing are five," and shows how this is wrong.

When the pupil sees that none are taken away, that the situation has not been disturbed, he realizes the fact at once. No further language is necessary. He does not need to be asked to say to himself, "nothing from five is how many?" The fact is so well known that it does not require oral or written demonstration.

Learning the expression, $5 - 0$, as learning to express a

thing that is so obvious as to need no expression.

Moreover, the writing of a 'zero combination' gives the appearance of expressing and describing an actual arrangement when none has been made. Suppose there are five objects in one hand and none in the other. Let the expres-

sion, 0 , be written. The expression, apparently describing an

arrangement of the objects that has been made, is altogether wrong when in fact no arrangement has been made. The pupil is asked to describe something that has been done with the group when it is perfectly clear that nothing has

been done. True, the pupil may learn to write 0 and to

say, "Five and nothing are five," but he will be learning to use a meaningless and misleading set of expressions which, though pretending to express something, express nothing.

When the pupil comes to dealing with tens and groups of tens, he will have no use for the zero except as a place holder. In his experience with tens, he may find what appears to be a 'zero combination' in one or more of the combinations. For example, in the additions shown it appears that a 'zero combination' must be dealt with in the tens column. In both, the zero is first observed and then explained.

first, one notices five represented in the unit's column, and that is all he finds represented. In setting down below the line the number represented, he sets down what he sees. In the second, one notices five and three represented in the unit's column, so five and three are thought together as eight. The zero is neglected because it is something that does not influence the arrangement to be performed or to be thought. The zero, being used as a place holder, is attended to as a place holder.

Zero combinations have been giving pupils unusual difficulty for the past twenty years. About twenty years ago, they began to appear in tests, and pupils were confused by them. The remedy used for the difficulty was that of 'teaching' these zero combinations, or trying to teach them. Teachers have tried to explain them, but, since they mean nothing, explanations have added to the confusion. The remedy suggested here is to omit them, both from the teaching and the testing, since they are both meaningless and useless. Their uselessness will be indicated from time to time in connection with the discussions of operations which *apparently*, though not actually, employ them.

VI. THE IMPORTANCE OF TEN

The importance of the idea of ten has been indicated. Its importance is unique. No other group is used in the same sense. When one is dealing with groups smaller than ten, he deals with separate groups. When he deals with groups larger than nine, he deals with them as tens. Everything beyond nine is stated as ten, or some grouping or arrangement of tens.

In expressing the ideas beyond nine in writing, one learns to resort to the use of position. In one position a figure represents units; in another, tens; and so on.

In number-thinking, one thinks of groups to ten and then thinks of groups by tens in exactly the same way.

The idea of ten is an idea of paramount importance: every

activity that can be used to get pupils to develop a number ten used. As the pupil makes progress in his activities he will make understandable progress in the subject that he is familiar with ten and is conscious of the use and importance. Each succeeding activity, up to and through the study of decimals and percentages, will require the pupil to enlarge his idea of ten and to depend upon the position of the figures he writes as he thinks, and each succeeding activity may serve to enlarge his idea of ten and his use of position. The beginner cannot learn at the outset all he will need to know about ten and about position. He will have the chance to learn more and more about them as he enlarges them in succeeding activities. What is important now is that he be made conscious of the ideas of ten and of position, so that he will have something in his own mind to rely upon as he moves along to succeeding stages of his work.

VII. STUDYING THE GROUP OF TEN

The work begins with a systematic study of the group of ten. The object is to enlarge and clarify the pupil's idea of ten. The methods are the same as the ones the pupil has learned to employ in studying the groups of ones.

1. Enlarge the Idea of Ten by Comparisons

By means of the activities of comparing, such as are described in Chapter X, the pupils may enlarge their idea of ten. Let a group of ten be counted, and compared with a group of eight, seven, nine, five, six, etc.

When actual comparisons have been made and described by the oral and written language of comparison, let the pupils answer such questions as:

"Ten is how many more than eight?"

20

8

"Ten is how many more than seven?"

20

7

"Ten is how many more than nine?"

20

9

2. Take Apart and Put Together Arrangements of Ten

With a group of ten objects, say, pencils, before each of the pupils, the teacher asks such questions and gets such replies as these:

"How many pencils have you?"

"Take three away. How many are left?"

"Tell what you have done." "Three from ten is seven."

"Write what you have done."
$$\begin{array}{r} 10 \\ -3 \\ \hline 7 \end{array}$$

"Three and seven are how many?"
$$\begin{array}{r} 3 \\ +7 \\ \hline 10 \end{array}$$

By similar questions, the attention of the pupils is called to the various arrangements that can be made with ten objects, and the pupils learn to describe each arrangement both orally and in writing.

Following are the written expressions of the arrangements of ten that the pupils learn to make and to describe:

$\begin{array}{r} 10 \\ -9 \\ \hline 1 \end{array}$	$\begin{array}{r} 10 \\ -8 \\ \hline 2 \end{array}$	$\begin{array}{r} 10 \\ -7 \\ \hline 3 \end{array}$	$\begin{array}{r} 10 \\ -6 \\ \hline 4 \end{array}$	$\begin{array}{r} 10 \\ -5 \\ \hline 5 \end{array}$	$\begin{array}{r} 10 \\ -4 \\ \hline 6 \end{array}$	$\begin{array}{r} 10 \\ -3 \\ \hline 7 \end{array}$	$\begin{array}{r} 10 \\ -2 \\ \hline 8 \end{array}$	$\begin{array}{r} 10 \\ -1 \\ \hline 9 \end{array}$
$\begin{array}{r} 9 \\ +1 \\ \hline 10 \end{array}$	$\begin{array}{r} 8 \\ +2 \\ \hline 10 \end{array}$	$\begin{array}{r} 7 \\ +3 \\ \hline 10 \end{array}$	$\begin{array}{r} 6 \\ +4 \\ \hline 10 \end{array}$	$\begin{array}{r} 5 \\ +5 \\ \hline 10 \end{array}$	$\begin{array}{r} 4 \\ +6 \\ \hline 10 \end{array}$	$\begin{array}{r} 3 \\ +7 \\ \hline 10 \end{array}$	$\begin{array}{r} 2 \\ +8 \\ \hline 10 \end{array}$	$\begin{array}{r} 1 \\ +9 \\ \hline 10 \end{array}$
		$\begin{array}{r} 5 \\ 2 \overline{)10} \\ \hline 10 \end{array}$			$\begin{array}{r} 2 \\ 5 \overline{)10} \\ \hline 10 \end{array}$			

It will be noted that there are nine new arrangements of subtraction and nine new arrangements of addition. These, added to the ones the pupils have already learned, make a total of 45 arrangements of each.

Practice continues by having the pupils *think* the arrangements in answer to the teacher's questions, by having them tell and write 'the whole story' of ten, etc. Finally, the

arrangements are practiced as they are made in groups in chance order. The exercises for practice are constructed as described in the final section of the preceding chapter. These include the 45 combinations of addition and the 42 combinations of subtraction.

The teacher should not neglect suggesting the pupils to give attention to arrangements of ten objects where the objects are present only in the imagination. Problems such as those suggested in the preceding chapter should be considered.

"Jimmy had 10 pieces of candy. He gave away 6 pieces. How many pieces of candy had he left?" etc.

"Sarah has 7 red stamps and 3 blue stamps. How many stamps does Sarah have?" etc.

"Janie picked 10 roses. She had three bouquets with some roses, putting 2 roses together in each bouquet. How many bouquets did she have?" etc.

As in the activities suggested in the preceding chapter, here let the pupils each in turn suggest 'problems' dealing with ten. The objects in the room, their representations at home and on the playground, lying at the teacher's disposal, provide the materials for many 'problems' that both they and the teacher can give. It will be well for the teacher to give 'problems' dealing with ten, to give problems involving eight, seven, and so on.

3. Study Ten in Combination to Fifteen

Through the study of the various possible arrangements of the group of ten and through the activities of investigating the group of ten with smaller groups, the group has developed a fairly definite notion of ten as a group, and the symbol, 10, has acquired more definite meaning. The pupils now need to learn more about the group, because it is larger than their previous practice in counting ten objects; and, at the same time, they need to learn some ideas about ten and its use. The first step is the development of a series

understanding of the relation between the idea of ten and the ideas, eleven to nineteen. Attention should be called to the relation, both as it is demonstrated by the teacher and as the manner of expressing the ideas in writing makes use of and illustrates the relation.

4. Emphasize the Group

Illustrate by the grouping of objects, first, the fact that groups of objects smaller than ten are each thought of as a single group; and second, the fact that groups of ten and of more than ten are each thought of as a group of ten and so many more, as the case may be.

Show a group of five, or seven, or eight, or nine objects. Point out that, when counted as five, let us say, they are put together or thought together all in one group. Put the objects together in one group. Point out that the group is expressed by one figure in writing, 5. Make the same illustrations again and again with other groups to nine, inclusive. Show that each is put together or thought together as one group, and expressed in writing by one figure: 1, 2, 3, 4, 5, 6, 7, 8, or 9.

Next, show a group of ten objects. Point out that they, too, are put together or thought together as one group. Point out that such a group is thought of as one group of ten, but that this group is expressed in writing in a different way. Point out that there is no figure, or sign, for ten, but that *the place where a figure is written* is intended to show ten in writing. Write 10 and compare it with the symbols 1 to 9. Show the place where 1 is written. Point out that the 1 is in ten's place and that it means, not 1, but 1 ten.

Point to the zero in 10 as something that puts the 1 in ten's place. Repeat the fact that to write ten, one has no new or different sign for it, as he has for the numbers one to nine, but has to make use of 1 written in ten's place. "I will write ten by writing 1 in ten's place." Write 1. Ask if that shows 1 ten. "What must I do to put the 1 in ten's

place, so that it will show 1 ten?" Let the answer be emphasized that the 0 is written to put the 1 in its proper place.

Next, show a group of eleven objects. Have the pupils count them. Point out that such a group is thought of, not as a single group with all the objects together, but as a group of ten and one more. Arrange the objects as a group of ten and one more. Point out that the number is to be thought of as ten and one, and that the number is expressed or written as one ten and one. Write 11. Point out that while it looks like two 1's, it does not mean two 1's. Emphasize the importance of position. Point out that 11 is written with 1 in unit's place to show one, and 1 in ten's place to show 1 ten; that 11 shows 1 and 1 ten. Referring back to the group of ten and one, or one and ten, called eleven, speak of the one as one, and of the ten as ten ten. Let these numbers be referred to again and again. Let the pupils write 11 and explain what each of the 1's stands for. Ask:

"What does 11 show?" "One and one ten, or one ten and one."

"What is 11 called?" "Eleven."

"Eleven is how many more than ten?"

"Ten and how many more are shown?"

Point out that the way eleven is written shows that eleven is one more than ten, and that ten and one are shown.

In similar manner, show groups of ten and two, ten and three, etc., to ten and nine, and explain what the symbols 12 to 19 show in each case.

Let this work be reviewed again and again.

"Write nine and ten together." "19."

"What does 19 show?" "Nine and one ten, or one ten and nine."

"What is 19 called?" "Nineteen."

"Nineteen is how many more than ten?"

"Ten and how many are shown?"

"In the figures 19, what does the 9 show?" "What does the 1 show?" "Why does the 1 in 19 show 1 ten?" "Write 1 ten by itself" "10" "What is the zero for?" and so on

Let the practice continue until the pupils are able to see in 15, for example, not a symbol that represents fifteen objects all in one group, but a symbol that represents fifteen objects grouped as one ten and five.

5. Groups of Tens

In order further to impress the fact that in arithmetic one must think in terms of ten, the pupils should be introduced to the method of thinking in groups of tens, to the system of writing the symbols, 20 to 100, and to the significance of these symbols as written. This will include some counting, some writing and reading of the symbols, and some demonstration of the groups for which the symbols stand. It should not be necessary to explain the meaning of every set of symbols from 20 to 100 or to demonstrate objectively every idea represented by them.

It is to be assumed that the pupils, through the various objective demonstrations and activities previously described, have already developed some fairly definite notions of the group of ten and of thinking of numbers beyond ten in relation to ten. The pupils should now be given an opportunity to extend these ideas, to enlarge upon them through their active employment. It is possible to demonstrate every idea from ten to one hundred through the use of objects in groups of tens. It should be kept in mind, however, that while some demonstration is necessary to set the pupils to thinking, continued demonstration may retard their progress in thinking. One advantage of the system of using the symbols the pupils are now to learn is that, through emphasizing a common method of grouping, the system makes continued use of detailed representation unnecessary. Enough demonstration needs to be provided by the teacher to give the

pupils the idea of method of grouping. When the pupils have developed the idea, they should be encouraged to choose the method of grouping that is used.

6. Use Objects in Groups of Ten

Toothpicks are cheap and easy to handle for the demonstration. Count out ten groups of ten each and slip a small rubber band around each group of ten to hold them together. Do this before class to save time. At the class assembly, call attention to the bundles of toothpicks. Give each pupil a bunch, or have them pass from pupil to pupil, with instructions to count the toothpicks in each bunch. Let the pupils repeat the number ten as each bunch as it is handed back.

"How many bunches of toothpicks have I?" Ten bunches.

"How many toothpicks in the bunch?" Ten. Write 10.

"Here are two bunches of toothpicks, ten here and ten here are two tens. This is the way to write two tens, 20."

"Here are three bunches, ten here, ten here and ten here are three tens. This is the way to write three tens, 30."

"Here are four bunches " " and so on, until all the numbers from 10 to 100 are written.

Hold up five bunches. Ask how many. Have a pupil point to the figures that show five tens, 50. Find out what the 5 in 50 is in ten's place, and show 5 tens. Repeat with six bunches, nine bunches, two bunches, etc. When the pupils have clearly in their minds that 50 shows five tens, 10 shows one ten, and so on.

Summarize by showing how the toothpicks are counted as one ten, two tens, and so on to ten tens.

Now, teach the names of two tens, three tens, etc. If the pupils do not already know these names, say them out loud to one hundred. When the pupils have learned the names, have them count the hundred toothpicks ten by ten, twenty, thirty, and so on.

Give the pupils practice in associating the symbols both with their names and with what they stand for:

10 is called ten; it means 1 ten.

20 is called twenty; it means 2 tens.

90 is called ninety; it means 9 tens.

7. Review Groups Already Learned

For the sake of proper associations, let the numbers ten to nineteen be reviewed. Have the symbols written and their meanings discussed. For example, let the children point out that 15 shows 1 ten and five, that 10 shows 1 ten, and so on. Do not have a written expression until the children have given it its common name; for example, 15, fifteen; 10, ten, and so on. What each expression actually shows should always be associated with what it is commonly called.

8. Lead to an Extension of the Idea

Suppose the discussion is upon 16. The pupils call it sixteen, and point out that it shows 1 ten and 6. Substitute 2 for the 1 in ten's place.

"How many tens do the figures show?" "Two tens."

"What is the whole number shown?" "Two tens and six."

"What is 26 called?" "Twenty-six."

If the pupils have difficulty in arriving at the name, twenty-six, point to the symbol, 20, which should be before the pupils on the board. Ask, "how many tens?" and when the answer is given, get the name, twenty. Return to the figure, 26, and give it its name. (It is to be kept in mind that the important thing for the pupils to get in mind is the significance of 26; namely, two tens and six. The name, twenty-six, can be fixed in the exercises of counting or otherwise.)

It may be well to vary the exercise just described. If the original discussion is upon 16, substitute 0 for 6. Have the

figure read, and its value told. Now substitute another sign for the 1 in ten's place; for example 2. How many tens are shown? ("Two tens.") "What is that called?" ("Twenty.") In like manner, substitute other symbols in ten's place, until the pupils recognize 30, 40, 50, etc., not merely as thirty, forty, fifty, etc., but also as showing three tens, four tens, five tens, etc.

Or, demonstrate the idea, twenty-six, by two bunches of toothpicks and six loose ones. Show the ten bundle. "How many toothpicks?" ("Twenty," or "two tens.") Write 20, to show twenty, or two tens. Place out that the 2 is in ten's place to show two tens. Ask what the sum is for.

Now, show the two bunches and the six loose picks.

"How many here?" "Twenty," or "two tens."

"How many are here?" "Six."

"How many do I have altogether?" "Twenty and six," or "twenty-six."

"This is the way to write twenty-six. I write two (2) in ten's place, and two (2) in ten's place - (20). Then I write six tens and six, or twenty-six."

"When I write twenty, I write two (2) in ten's place. Do I need a zero (0)? Why?"

"When I write twenty-six, I write two (2) in ten's place and six (6) in unit's place. Do I need to write a zero (0)? Why not?"

In like manner, other numbers are demonstrated and expressed in writing.

9. Exercises

Exercises such as are indicated by the following lines should now be conducted until the pupils have fixed the idea of ten's place and of the significance of numbers in ten's place.

28 is — tens and — (units or ones understood, but not necessarily named)

78 is — tens and —

As this number chart develops, it should be reviewed in frequent reviews. The teacher may point to a number 47, for example. The pupils should give the names forty seven, and tell the meaning, namely, it is 4 tens and 7

11. Point Out the Significance of the Zero

While it is not to be expected that pupils at the stage of development represented by the work space having described will be able to arrive at a complete understanding of the significance of the zero, something of its importance may be pointed out to them. The zero is not a symbol for a quantity. Its use is merely to hold a position. When it is written, the 2 represents 2 tens because it is in the tens place. It is the second position to the left, because the 1 is in the (or one's) place holds it there. Now, if we have no number to write, we must have some sign to take the place of a number in order that the 2 may be held in the second or tens position.

No elaborate explanation of the use of the zero is recommended. The significance may be described as given as the pupils have impressed upon them, through the previous activities described in the foregoing pages. The teacher should 25 is 2 tens and 5 and that 20 is 2 tens and 0 or 20 is 2 tens and 0, but 1 and 1 ten or 1 ten and 1. As the work progresses, the teacher should indicate that 10 is 1 ten and 0 or 1 ten and 0 and so on.

12. Emphasize the Ten's Position

Coincident with the pointing out of the use of the zero, as opportunity affords, the place where the zero is written should be emphasized. Thus the pupils have been directed to their attention indirectly in all the ways that have been suggested in this chapter. The teacher should take advantage of every opportunity to call the matter to the pupils' attention. This should be only in connection with the regular work on the number chart.

to the development of the idea of tens and (let us repeat) only as opportunity affords. As with the zero, this particular idea of form of written expression of number ideas should develop only in connection with the development of the number ideas.

The idea of ten, the significance of the zero, and the importance of position will be called into constant and ever-increasing use as the pupil moves forward through the various steps of arithmetic. The general ideas named develop gradually; but as they develop, they throw light upon, and make intelligible the many phases of the various processes with whole numbers and fractions that the pupil will be called upon to learn. Moreover, they will serve, to the extent that he develops them, to knit together into a consistent scheme, or system, all the various processes that at first glance appear to be so different each from the other. Thus, what the pupil is beginning to learn now will, when extended and enlarged and clarified in succeeding grades, take him through the later stages of his learning up to, and including, decimals and percentage.

VIII. SUMMARY

By means of the activities described in this chapter, pupils first become acquainted with the group of ten in its relations to smaller groups; and second, they begin that development of the idea of ten that may serve to throw light upon and to make clear all subsequent work in arithmetic. The present activities reveal to pupils for the first time something of the unique importance of the group of ten as a standard by which all other groups are evaluated. The activities give pupils their first introduction to the number system they must learn.

CHAPTER XIII

EXTENDING THE NUMBER IDEAS

AIMS

1. Through objective demonstration and practice pupils may learn that tens are added and subtracted just as units are added and subtracted.
2. Such demonstrations and practice give opportunity for further emphasis upon the significance of tens in practice and the use of the zero.
3. They extend the practice on the simple additions and subtractions to higher levels.
4. Additions and subtractions in the higher decade series to extend practice to higher levels.
5. The relations to the simpler additions and subtractions should be made clear. The goal is understanding, not rote skill.
6. The higher-decade additions chosen for practice should be those that are useful in column additions and in multiplications.
7. The higher-decade subtractions are included mainly to extend the pupil's practice.
8. Column addition provides for an extension of practice to a new level.
9. The new type of addition should be understood before practice begins. To this end, objective demonstrations of the process is recommended.

I. REVIEW OF PREVIOUS STAGES

If the activities described in the foregoing chapters have been carried through thoughtfully and systematically, they have resulted in enlarging and clarifying of the children's number ideas to ten, and in attaching a special significance

of the idea of ten. In the development of these ideas, the children have studied groups and have given thoughtful attention to the various arrangements of these groups. Around each number idea as a center of reference, they have learned the arrangements, or combinations, that relate to it. They have learned in their relations 45 addition combinations and 45 subtraction combinations and have given some attention to 8 combinations of equal groups. Moreover, they have learned to deal with the idea of ten in a special manner, and to think of ten and of the method of grouping by tens as these give meanings to the numbers from ten to one hundred.

It will be well as a prelude to the activities suggested in the present chapter to take the pupils through a rapid review of what they have already learned. Such a review will be very useful, because the pupils will be called upon, in undertaking the activities now to be suggested, to make use in ways that are new and different of the ideas they have developed and of the arrangements they have already learned. The review will make easier their introduction to the new activities. The new activities, in turn, will provide new associations for the arrangements the pupils have learned and new uses for the ideas they have acquired. The new activities will be discovered to be not entirely new and different, but in reality means of extending and enlarging the uses of what the pupils know. The new activities will thus provide a new and different kind of review — in a sense, a new view of things already familiar.

II. ADDING AND SUBTRACTING TENS

1. Purpose of the Activities

The purpose of the activities described in this section is to introduce the pupils to the idea that tens may be dealt with in exactly the same way as units. As the pupils gain this idea slowly and gradually, they will slowly and gradu-

ally discover that they are in possession of the key that serves to unlock the mystery of *known* arithmetic. They will discover, as they make progress in understanding arithmetic, that tens are multiplied just like units and divided just like units, that one multiplied by tens and divided by tens gives us the multiples and divides by units, that hundredths, tenths, hundredths, etc., are added, subtracted, multiplied, and divided just like units, and so on. As they move forward to the more and more complex processes of arithmetic, which in comparison with the simpler processes are more and more difficult, they may find themselves in greater and better possession of the idea that *known* arithmetic is a complex process and make their understanding of it. Then, by gaining possession of the idea that will make the *known* processes easy to understand, the pupils may find as they proceed that the apparently more and more difficult processes may be attacked by them more and more effectively and successfully. And thus, instead of requiring as we have more explanation and direction as they move forward to the later complex processes of arithmetic the pupils will be able to get along with less and less explanation and direction. By being treated in this way the pupils will be able to rely upon their gradually developing ideas for guidance in the newer processes to them and to give them *known* ideas how to proceed.

Specifically, the present section will show how to introduce the pupils to the fact that tens are divided as well as units by instructing them and giving them problems in the addition and subtraction of tens. This may be done by adding and subtract units to the extent of tens and then by adding together and take away groups relating to the units and tens and having been introduced to the idea of tens and the importance in connection with numbers large than ten the pupils may now proceed to the processes of adding tens and subtracting tens.

2 Use Groups of Objects to Demonstrate

For the sake of illustrating what the pupils have to learn, the teacher may resort to the use of toothpicks in bunches of ten, as were used in the demonstrations of the preceding chapter.

Pass around several bunches. Have the children count the picks in each bunch, so that they will be clear that there are ten in each.

Place the bunches before the pupils and ask: "How many bunches are here?" "Eight," let us say.

Remind the pupils that each bunch is a bunch of ten.

"How many picks are here?" "Eighty, or eight tens."

"Let us write eight tens, or eighty."

80

"Now watch as I take away five tens, or fifty."

-50

Write 50 with the minus sign (-) before it, and draw a line to show that fifty, or five tens, is to be taken away from eighty, or eight tens.

30

"How many are left?" "Three tens, or thirty."

Write 30 in its proper place.

Review the activity with eight picks, taking five away and leaving three. Repeat the activity with eight bunches (tens), taking away five bunches (tens) and leaving three bunches (tens). Emphasize the similarity between "five from eight is three" and

$$\begin{array}{r} 8 \quad 80 \\ \text{"five tens from eight tens are three tens." } -5, \quad -50. \\ \hline 3 \quad 30 \end{array}$$

Make impressive the point that tens are subtracted just like units. Again state the question, "Fifty from eighty is how many?"

80

in writing, -50. Point out that the 8 is in ten's place, and means 8 tens, that the 5 is in ten's place, and means 5 tens. Subtract. The pupils know that the answer will be 3 tens, but let it be emphasized. Show that the subtraction is just the same as "five from eight is three," but that the 3, since it means 3 tens, must be writ-

80

ten in ten's place. Write the 3 in the answer, -50.

3

"What is wrong with the answer? I have subtracted, 'five from

eight is three,' but I want the 3 to show 3 tens. What must I do?" "You must write a zero (0) in both the ones & tens place and put the 3 in ten's place."

Write the zero in the proper place.

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Point out, when such a question is given, that one realizes that he has to take away tens, and he knows ahead of time that the answer must be written in ten's place. In an order to put the answer in ten's place, he writes a zero (0) in the hundred's place, then goes ahead and subtracts tens just like usual. "Five tens, eight is three" — and writes the 3 where it belongs.

Let the pupils practice a number of subtractions of the tens and on their papers. Let them explain why they write the zero (0) first, and how they subtract the tens.

$$\begin{array}{r} 60 \\ -40 \\ \hline 20 \end{array} \quad \begin{array}{r} 50 \\ -30 \\ \hline 20 \end{array} \quad \begin{array}{r} 80 \\ -50 \\ \hline 30 \end{array} \quad \begin{array}{r} 90 \\ -60 \\ \hline 30 \end{array} \quad \begin{array}{r} 70 \\ -40 \\ \hline 30 \end{array} \quad \begin{array}{r} 60 \\ -30 \\ \hline 30 \end{array} \quad \begin{array}{r} 70 \\ -40 \\ \hline 30 \end{array}$$

Return to the bunches of sticks for further demonstration — five bunches and three bunches, for example. In a manner similar to that used in demonstrating the subtraction of tens, illustrate the addition of tens. Encourage the pupils to think of the five bunches and of the three bunches, as five tens and three tens, and to write them as 50 and 30.

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"How many are fifty and thirty?" 20

Let the pupils count the five bunches and the three bunches together to get eight bunches, and demonstrate the fact that in adding 50 and 30, one adds the tens just like usual. "Five and three are eight," and writes the eight (8) in ten's place to show eight tens, or eighty.

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Point out, when such a question is asked as, 20, that one realizes that he has to add tens, and he knows ahead of time that the answer must be written in ten's place. In an order to put the answer in ten's place, he writes a zero (0) in the hundred's place, then goes ahead and adds the tens just like usual. "Five and three are eight," and writes the 8 where it belongs.

Let the pupils practice a number of additions of the tens and

on their papers. Let them explain why they write the zero (0) first, and how they add the tens.

40	30	50	60	30	20	10
<u>50</u> ,	<u>70</u> ,	<u>20</u> ,	<u>40</u> ,	<u>20</u> ,	<u>70</u> ,	<u>50</u> , etc.

Illustrate the performance of the operations in such combinations as $78 - 40$, as follows:

Let the 78 be represented by seven bunches of picks and eight loose ones. Point out that the first operation, "six from eight," means six picks taken from the eight. Let them be taken away and two are left. Write 2

Point out that the second operation, "four from seven," means four tens from seven tens. Let four bunches be taken away, saying, "Four from seven is three." Three (tens) are left. Write 3 (in ten's place). Since 2 has already been written, it will put the 3 (tens) in its proper place.

Illustrate the performance of the operations in such combinations as, $32 + 40$, as follows:

Let the 40 and the 32 be represented by four bunches of picks and six single ones, and three bunches and two single ones, respectively. Point out that one first adds the units, "six and two are eight," illustrating by putting together and counting together the six picks and the two picks. Write 8

Point out that the next operation "four and three" means four (tens) and three (tens), and that one adds, "four and three are seven," and knows that the answer is seven tens. Illustrate by putting together and counting together the four bunches and the three bunches. Write 7 (tens) in its proper place. The sign previously written (8) puts the 7 in ten's place where it belongs.

3. Teach the Pupils to Extend Their Ideas

Objective demonstration of thinking groups of ten together and thinking them apart should be continued until the pupils get the idea that tens are added and subtracted just like units. The purpose of the demonstrations is to get the pupils to thinking about the arrangements of groups

just as they have already learned how to arrange the arrangements of objects. When the purpose has been accomplished, the demonstration should cease and the child should move ahead in the direction of using the knowledge they have gained in performing the work.

Questions requiring the arranging of items should be asked and answered both orally and in writing. The child should be able to answer the questions, "Twenty and thirty are

20

how many?" and 30, without using the fingers. When a child need to have sufficient practice in writing the answers to the additions and subtractions of items to understand the written expressions with their fingers and fingers.

Questions requiring the arranging of items should be asked and answered in writing. The child should be able to

25

75

43 and 44, need to be written down. The child should be able to do steps in thinking. The purpose of the work is to do the work as units, and that child should understand the work as units. When the units have been learned, the child should be able to do the work according to the direction the question gives. The child should be able to do the work, because the child should be able to hold it in memory while they perform the work.

asked for. Having learned the work, the child should be able to do the work and to do the work in writing. The child should be able to do the work with equal ease to add, 33, and to subtract 33.

with equal ease to add, 33, and to subtract 33. The child can quickly see that the work is the same as the work with the units.

4. Emphasize the Use of Zero.

In the addition and subtraction of items, the zero appears in calculations with negative numbers. Attention should be called to the zero, and the child should be able to do the work with a group that may be arranged with the zero.

sign that *holds a place* and keeps the positions of the numerals clear in the written expression of number ideas.

Some illustration of the manner of presenting the use of the zero has already been given.

25 78

In combinations like the following, 40 and -50 , show, in the case of the unit's column that one has five in one group, but does not add anything to it; or that one has eight in one group, but does not take anything from it; and that one still has five in the one group and eight in the other. So starting with five or eight, and letting the five alone and the eight alone, one still has five in the one example and eight in the other. So 5 is written as the answer to the one and 8 as the answer to the other.

Since the zero (0) can be considered as a symbol for quantity, only in a sense that is highly abstract, do not undertake to call attention to what are sometimes called the 'zero combinations.' In the experience of the child they do not exist; and in the experience of the adult, the zero, as such, in addition and subtraction, is neglected. The zero, being a mere *place holder* in our system of notation, does not affect the sum or difference one way or the other, and it should not be called to the attention of children, in so-called 'zero combinations,' as though it did.

84 78

In combinations like the following, -54 , -28 , etc., point out in each case, starting with four and taking four away, etc., that no units are left, and no figure to show units is to be written; but since tens are next to be taken from tens, leaving tens for an answer, the answer must be *written in ten's place* in order to show tens. Therefore, in order to place the answer, when one gets it, where it belongs, one first writes a zero (0) in unit's place to put the answer where it belongs. Thus, the pupils may subtract: "Four from four is nothing." (There is no number figure, no numeral, to be written, so one writes 0 to make the next figure show

tens). "Five from eight is three." The three is written '3' under the 5 (tens) and 8 (tens) in above 2 (tens) tens. The zero (0) already written makes the 3 show three tens.

5 Practice Exercises

In the practice exercises that should follow, no carrying or addition or subtraction should be introduced in the exercises that use the 36 addition and the 36 subtraction combinations relating to the idea of ten, such as, 22

24 26

etc., no carrying is possible. The practice should involve the use of the 9 addition and the 9 subtraction combinations that relate to the idea of ten. Carrying may be introduced by taking care to place all such combinations in the tens + ones

43 36 108 105

sums, thus: 62, 71, 85, 93, etc.

The practice of the pupils should be systematic. It should be practice in thinking as well as practice in written expression. In the beginning the work should be slow and deliberate. Let the pupil "think aloud," explaining step by step what he is doing or, when he has finished an exercise, let him explain step by step what he has done. Let the fact that tens are added and subtracted just like units be repeated again and again; and let the pupils explain again and again the use of the zero in hold position. Finally when the pupils are perfectly clear in their own minds as to what these words mean, the practice may be speeded up by a gradual reduction of the "thinking out loud" and of the rapidness required. Finally, the pupils should add or subtract in small column in exactly the same way just like units and read their answers by giving the usual names, thus, fifty, twenty-seven, sixty-five, eighty, etc.

6 Summary

The activities that have been described in this section are the activities of using what the pupils have learned in the

gaining of new knowledge and understanding. The various arrangements of addition and subtraction that relate to the ideas to ten are used in both old and new ways. Combined with the idea of ten, whose development had already begun, these arrangements have been used as arrangements of units and arrangements of tens. The pupils have learned that tens are added and subtracted "just like units," they have learned to put some reliance upon *position* as an aid in thinking, and they have gained practice in simple arrangements. They may now proceed to further practice of the simple arrangements as they appear in new forms and with new uses.

III. ADDITIONS AND SUBTRACTIONS IN THE HIGHER DECADES

1. Results of Previous Activities

By means of their previous activities the children have developed somewhat definite ideas of groups to ten, together with some notion of the special importance of the group of ten. They have learned the various arrangements that build up and relate to the ideas mentioned, and they have learned special uses of these arrangements in connection with the activity of grouping by tens or of thinking objects as arranged in groups of tens. They have practiced the arrangements mentioned, both as simple arrangements and as arrangements of groups of tens. The pupils are now ready to gain further practice with the arrangements they know in connection with a further extension of their ideas.

2. Two Kinds of Practice

The teacher of beginners does not expect mastery all at once, but the teacher does expect that the pupils will make progress in the direction of mastery. In order that the pupils may finally master the arrangements they have learned,

8
7

15. In the column 6, the result of adding 8 and 7, 15, is added to 6. The first form, 15 and 4, does not involve 'bridging' to the next decade. The latter form, 15 and 6, does involve 'bridging.'

43

In the multiplication, 8, for example, one first secures the product, 8×3 . He next secures the product, 8×4 , and adds the 2, which is 'carried' from the former product. The answer, 34, is in the same decade with 32. In the mul-

85

tification 6, the second product, 48, is added to the 3, which is 'carried' from the first. The answer, 51, is in the next higher decade. In the former case, no 'bridging' is involved; in the latter case, 'bridging' is needed.

Higher-decade addition without bridging must be distinguished from higher-decade addition with bridging. In the present chapter, the exercises will be confined to examples of the former type, with one exception. The pupils have

9 8 3 4

learned such combinations as, 1, 2, 7, 6, etc. They

10 10 10 10

should be given opportunity to extend such combinations

19 29 39

18 28 38

13 23 33

to 1, 1, 1, etc., and 2, 2, 2, etc., 7, 7, 7, etc., and

14 24 34

6, 6, 6, etc.

a. *The Extent of Higher-Decade Addition.* There are, all told, 765 separate higher-decade addition facts from 10 plus 1 up to sums of 99. To give specific and intensive drills upon each of these facts would make the task of *fixing* the idea a well-nigh insurmountable one. However, as preceding topics have indicated, there are certain of these 765 combinations that pupils will have special need for in column addition and in multiplication. These, it will be seen, are few enough in

The additions of common occurrence, which total 161 without bridging and 125 with bridging are the ones that need to be included in the special drills. In the present chapter the 161 additions that do not involve bridging are considered, and attention is given to the higher-decade addi-

18 29 36 54 63

tions relating to the idea ten, such as 2, 1, 4, 6, 7, etc. Exercises including these additions should be given special attention.

b. Extending the Children's Ideas Higher-decade addition is not simple addition combined with "bring down the next figure." It is an *extension* of simple addition — an exercise that involves the development of a general notion of the *application* of simple addition to a situation that is differ-

25

ent. For example, in adding 25 and 4, 4, the pupil is not to add $5 + 4$, 9, and then 'bring down' the 2 to its proper place in the total result. He must learn to add in a single operation, thus, "25 and 4 are 29." In both column addition and multiplication there is no chance in effective work to add 25 and 4 in two separate operations; 4 must be added to the 25 in one operation.

It is commonly believed that, once the pupils learn the simple combinations, they are able to apply them in the higher decades without any special training. It is true that many pupils do learn to generalize and to apply their experiences without guidance or outside direction, and that in the course of time most pupils will learn to generalize and apply by haphazard methods. On the other hand, it is well proved that pupils learn to generalize and apply their knowledge earlier and more effectively when they are trained to generalize.

c. Understanding, Not Mere Skill, as the Goal. Specifically, the purpose of the present section is to suggest means of helping the pupils to build up a single general idea, and not merely to suggest practice exercises on the 161 useful higher-

may be used. They all are different, to be sure; but they all are very much alike, and that is the important thing to consider. Attention must be given to enough of the additions in the higher decades to become familiar with their *common characteristics*, their relations to the simple additions; when these are discovered and observed by the pupils, they will be able to deal with higher-decade additions whenever and wherever they find them.

When the pupils have developed the idea of similarity or of relationship, they need practice in using the idea so that it will be ready for use when needed. Abundant practice, when it is delayed until the pupils understand, is always valuable. The inclusion of the 161 'useful' additions in the practice exercises provides such abundant practice. In the practice of the idea, a less useful addition is just about as valuable for practice as a more useful one; if any choice is to be made, however, the more useful ones should be favored. Concession must always be made to the fact that practice upon the more useful is better than practice upon the less useful, if other things are equal. The purpose of practice, however, should not be lost from sight. In the present case, the purpose is to practice using an idea *after* it has been gained.

Let us turn now to a consideration of how the idea in question may be developed.

d. Illustrating the Processes. For the sake of illustration the teacher may resort to the use of toothpicks bound together in bunches, or groups, of ten, supplemented with a few loose picks. The process might then be developed like this:

Recall a few of the familiar addition arrangements, such as,

$$\begin{array}{cccc} 2 & 6 & 4 & 2 \\ 3 & 3 & 3 & 6, \text{ etc.}, \end{array}$$
 by mentioning, writing, and illustrating them with

$$\begin{array}{cccc} \overline{5} & \overline{9} & \overline{7} & \overline{8} \end{array}$$

the grouping together of the loose picks.

Next, place before the pupils a bunch of ten and two. Let the

ADDITIONS AND SUBTRACTIONS

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pupils name the number. *Teacher* and *pupils* are in a position to do this.

Inquire as to what 12 shows *one ten and two*

Now, ask the question, "How many are twelve and one?" *13*

12

the question, 6. Illustrate the grouping of twelve and one, so that the pupils can give the answer at once. Let the pupils take a group of six sticks that are added to the twelve (ten and two) and then count all the sticks. (Let us insist that the group of ten has already been counted and is known to be of value ten. The pupils count the two and six together and determine what parts make

13

the group of ten. Let the answer now be given and written.

13

12

Let the arrangement, 6, be written beside the simple arrangement

13

2 12

ment, thus: 6, 6.

8 18

Ask the pupils if our looks something like the other. Let them

12

tell how the actual arrangement, 6, is like ten and two and one

13

2

compares with the actual arrangement, 6 (10 and one)

13

22 22 22 22

2

Next, illustrate the relation of 6 6 6 6 and so on, in the same way. Call on the pupils to take in the other arrangements.

Illustrate, and let the pupils illustrate, how the other simple such relations in the same way, for example:

4 14 24

5 15 25

3, 3, 3, etc.

4, 4, 4, etc., etc.

7 17 27

8 18 28

When the pupils have learned to give attention to the idea of relationship between the simple arrangement and the other simple arrangements by means of the illustrations let the teacher be pointed out and attended to by means of the simple arrangements of the arrangements only. Hence such, especially in the beginning

$\begin{array}{r} 3 \\ 5 \end{array}$ $\begin{array}{r} 13 \\ 5 \end{array}$ $\begin{array}{r} 23 \\ 5 \end{array}$, and so on to $\begin{array}{r} 93 \\ 5 \end{array}$, $\begin{array}{r} 4 \\ 2 \end{array}$, $\begin{array}{r} 14 \\ 2 \end{array}$, $\begin{array}{r} 24 \\ 2 \end{array}$, and so on to $\begin{array}{r} 94 \\ 2 \end{array}$.

Let the pupils in turn give the answer orally.

Let the pupils in turn write the answers to a related set, and give them orally.

Let the pupils in turn explain the extent to which all in a given set are related.

To carry the illustration further, recall and repeat two or three of the additions that the pupils have learned in relation to the

$$\begin{array}{r} 3 \\ 7 \end{array} \begin{array}{r} 6 \\ 4 \end{array} \begin{array}{r} 5 \\ 5 \end{array}$$

group of ten: $\begin{array}{r} 7 \\ 7 \end{array}$, $\begin{array}{r} 4 \\ 4 \end{array}$, $\begin{array}{r} 5 \\ 5 \end{array}$, etc. Let the fact be repeated and re-emphasized that ten objects are *thought together* as a group of ten.

Next, place before the pupils a bunch of ten and six. Let the name of the number be given, "sixteen," and expressed in writing, "16." Inquire as to what 16 shows "one ten and six."

Now ask the question, "How many are sixteen and four?"

16

Write the question, $\begin{array}{r} 4 \\ 16 \end{array}$. Illustrate the grouping. Let the pupils note a group of four added to the sixteen (ten and six). Let them count all the picks. (Bear in mind that the one group of ten has already been counted and is known as ten or one ten). The pupils count the six and the four together and determine another group of ten. For the sake of illustrating the fact that six and four are *thought together* as one ten, place them together into another bunch of ten. "How many tens are there?" Let the answer now be

10

given and written, "two tens," or "twenty," $\begin{array}{r} 4 \\ 20 \end{array}$. Let the arrange-

20

6 16

ment be written beside the earlier one: $\begin{array}{r} 4 \\ 10 \end{array}$, $\begin{array}{r} 4 \\ 20 \end{array}$. Ask the pupils if

10 20

one looks something like the other. Let them tell how the actual

16

arrangement, $\begin{array}{r} 4 \\ 20 \end{array}$, (one ten and six and four) compares with the ac-

20

6

tual arrangement, $\begin{array}{r} 4 \\ 10 \end{array}$, (six and four).

10

the class, read a row of the combinations with their appropriate answers. For example, let him read the first row of exercises, thus: "ten and two are twelve," "twenty and one are twenty-one," "eleven and three are fourteen," etc. The rest of the class will listen attentively and correct any error that may be made. When the pupil has finished the first row, let another pupil take the set of exercises, and read the combinations with appropriate answers from the second row, and so on.

4. Higher-Decade Subtractions

The purposes of teaching the higher-decade subtractions do not parallel those that obtain for the higher-decade additions. In the first place, there is no process of cumulative subtractions that correspond to the process of cumulative additions when a column is added. In the second place, short division, with its successive subtractions is comparable with the multiplication of a two-place (or larger) multiplicand, but in the program that this book is outlining, short division is not recommended as the predecessor of long division. As a consequence, there is no need for any such analysis of the higher-decade subtractions as was made of the higher-decade additions. The only purpose of teaching the subtractions is to give the pupils a larger view of the process of subtraction than they otherwise could have and also to provide for them an opportunity for the practice of the simple subtractions on a different level; in short, to extend their knowledge and understanding of subtraction.

a. Illustrating the Processes. For the sake of illustration let the teacher resort to the use of toothpicks bound together, as before, in bunches, or groups, of ten, supplemented with some loose picks.

Recall a few of the subtraction arrangements, such as

$$\begin{array}{r} 7 \\ -4 \\ \hline 3 \end{array}, \begin{array}{r} 9 \\ -3 \\ \hline 6 \end{array}, \begin{array}{r} 5 \\ -2 \\ \hline 3 \end{array}, \begin{array}{r} 6 \\ -4 \\ \hline 2 \end{array}, \text{ etc.,}$$

by mentioning, writing, and illustrating them with the loose picks.

ADDITION AND SUBTRACTION

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Next, place before the pupils a board of ten and seven. Let the pupils name the number, 'seventeen,' and remove it as nothing, "17." Inquire what 17 shows 'one ten and seven'

Now, ask the question, 'Fourteen from seventeen is how many?'

Write the question, $17 - 4 = ?$ Illustrate the withdrawal of ten and four from ten and seven. Let the pupils take the ten, and four as they are removed, leaving seven. Let the answer be given and written beside the corresponding statement of the thought arrangement.

$$\begin{array}{r} 7 \\ -4 \\ \hline 3 \end{array} \quad \begin{array}{r} 17 \\ -14 \\ \hline 3 \end{array} \quad \text{Point out the similarity}$$

Next, illustrate, and have the pupils illustrate, other similar arrangements, writing the result of each beside the original thing:

$$\begin{array}{r} 7 \\ -4 \\ \hline 3 \end{array} \quad \begin{array}{r} 17 \\ -14 \\ \hline 3 \end{array} \quad \begin{array}{r} 27 \\ -24 \\ \hline 3 \end{array} \quad \begin{array}{r} 37 \\ -34 \\ \hline 3 \end{array} \quad \text{etc.} \quad \text{Encourage the pupils to point out}$$

the similarity between the various similar arrangements.

In similar manner, let the pupils illustrate other subtraction arrangements, for example,

$$\begin{array}{r} 9 \\ -3 \\ \hline 6 \end{array} \quad \begin{array}{r} 19 \\ -13 \\ \hline 6 \end{array} \quad \begin{array}{r} 29 \\ -23 \\ \hline 6 \end{array} \quad \begin{array}{r} 39 \\ -33 \\ \hline 6 \end{array} \quad \text{etc.} \quad \begin{array}{r} 1 \\ -2 \\ \hline 3 \end{array} \quad \begin{array}{r} 11 \\ -12 \\ \hline 3 \end{array} \quad \begin{array}{r} 21 \\ -22 \\ \hline 3 \end{array} \quad \begin{array}{r} 31 \\ -32 \\ \hline 3 \end{array} \quad \text{etc.}$$

Place such questions on the board:

$$\begin{array}{r} 6 \\ -4 \\ \hline 2 \end{array} \quad \begin{array}{r} 16 \\ -14 \\ \hline 2 \end{array} \quad \begin{array}{r} 26 \\ -24 \\ \hline 2 \end{array} \quad \begin{array}{r} 36 \\ -34 \\ \hline 2 \end{array} \quad \text{etc.} \quad \begin{array}{r} 8 \\ -3 \\ \hline 5 \end{array} \quad \begin{array}{r} 18 \\ -13 \\ \hline 5 \end{array} \quad \begin{array}{r} 28 \\ -23 \\ \hline 5 \end{array} \quad \begin{array}{r} 38 \\ -33 \\ \hline 5 \end{array} \quad \text{etc.}$$

Let the pupils in turn give the answers orally.

Let the pupils in turn write the answers to a selected set and give them orally.

Let the pupils in turn explain the relation of each set of numbers to the others.

For further development recall and repeat the subtraction of ten or three of the subtractions that the pupils have learned in relation

$$\begin{array}{r} 10 \\ -3 \\ \hline 7 \end{array} \quad \begin{array}{r} 20 \\ -3 \\ \hline 17 \end{array} \quad \begin{array}{r} 30 \\ -3 \\ \hline 27 \end{array} \quad \text{etc.}$$

Next, place before the pupils two bunches of ten. Let the name of the combined groups be given, "twenty," and expressed in writing, "20." Illustrate the withdrawal of one ten and six from the two tens. Show the withdrawal of six from one of the tens, and of

20

ten from the other, leaving four remaining 16. In similar

4

manner, illustrate the withdrawal of twenty-six from thirty, thirty-six from forty, etc. Write the results in a set, showing relations, thus:

$$\begin{array}{r} 10 \\ -6 \\ \hline 4 \end{array} \quad \begin{array}{r} 20 \\ -16 \\ \hline 4 \end{array} \quad \begin{array}{r} 30 \\ -26 \\ \hline 4 \end{array} \quad \begin{array}{r} 40 \\ -36 \\ \hline 4 \end{array} \quad \begin{array}{r} 100 \\ -96 \\ \hline 4 \end{array} \text{ up to } \begin{array}{r} 100 \\ -94 \\ \hline 6 \end{array}$$

Let the pupils illustrate two or three other sets of arrangements in a similar way, for example,

$$\begin{array}{r} 10 \\ -4 \\ \hline 6 \end{array}, \begin{array}{r} 20 \\ -14 \\ \hline 6 \end{array}, \begin{array}{r} 30 \\ -24 \\ \hline 6 \end{array} \text{ up to } \begin{array}{r} 100 \\ -94 \\ \hline 6 \end{array} \text{ and } \begin{array}{r} 10 \\ -7 \\ \hline 3 \end{array}, \begin{array}{r} 20 \\ -17 \\ \hline 3 \end{array}, \begin{array}{r} 30 \\ -27 \\ \hline 3 \end{array} \text{ up to } \begin{array}{r} 100 \\ -97 \\ \hline 3 \end{array}$$

Let the rest of the sets of arrangements that relate to the simple ones belonging to the idea of ten be illustrated by reference to the written expressions. Let the pupils in turn give the answers orally, write the answers in a related set, and explain relations.

5. Summary

The activities of the present section teach the pupils to extend their knowledge of the simple additions and subtractions they have learned to the corresponding higher-decade additions and subtractions. The pupils thus receive not merely more practice upon what they have learned, but also practice upon a higher level.

IV. COLUMN ADDITION

1. Results of Previous Activities

The previous activities have led to the development of fairly definite and usable ideas of the groups to ten, and have introduced the pupils to the special meaning and use

of the idea of ten. They have had experience in a rehearsal of the idea of ten with the use of the appropriate blocks were worked out in the study of groups. In these studies the initial activities were those of making, or with regard to making, describing, and repeating the simple arrangements of objects; the later activities have been those of extending the knowledge of simple arrangements to the very numerous and related arrangements of groups of ten.

2 Purpose of the Activities

The purpose of the present activities, like that of the two immediately preceding, is intended to teach, or to help teach, and to teach the extension of the idea in the handling of the new. Or, to state the matter in another way, the purpose is to suggest a series of exercises that give the child some practice and a different kind of practice upon the old arrangements of addition that have been brought up in the past.

3 Two Kinds of Practice

The distinction between the two kinds of practice which have been made the subject of discussion in preceding sections is important enough to be repeated. There are two kinds: on the level of learning, and on the level of use. Both are necessary. The first makes ready for the second; the second serves the double purpose of practice and of the extension of experience.

More repetition may easily degenerate into meaningless repetition. Once the repetition gets under way it is an easy matter to neglect giving attention to the process of the activity being repeated. In the longer series of repetitions, or drills, attention can be kept upon the process of the activity by reference to the earlier activities that produced the results. Later, when these have become more familiar and more automatic, attention can be kept upon the process by varying the repetitions in such a way as to present a constant challenge to the attention.

Repetitions may be varied through the use of first one drill device then another. Reference has been made in Chapter XI to this method of providing variety. Repetitions may be varied by employing them in new, but related, activities. Adding and subtracting tens and the higher-decade additions and subtractions provide variety for the repetitions of the simple arrangements by carrying them to new, but related, activities.

4. How Column Addition is New

Column addition differs from simple addition, though it employs directly the arrangements of simple addition. In simple addition, the sum is set down as soon as it is determined. In column addition the sum is cumulative and is not recorded until the final addition is made. In simple addition, the expressions may be described as 'seen' expressions; in column addition, the expressions are both 'seen'

1

2

and 'unseen.' For example, in the column 4, 'one and two' is 'seen,' though the answer is not recorded; 'three and four' is 'unseen' because one of the addends is not visible.

1

2

The column 4 involves two simple additions that must be fairly well in hand before they can be applied in the new situation. Column addition involves holding in mind a partial sum and adding it to the next group, and so on, until the column is completed.

Column addition differs from simple addition in the respect that it requires for completion a longer span of atten-

1

2

tion. A column of two additions, such as 4, requires more than two additions. It requires the elimination of any rest period that may be possible when the pupil has to make two

the pupils discover by means of their study the meaning of the column, the teacher introduces the written form.

The purpose of the further study of groups to ten is not primarily to gain further ideas of the groups, although such may be a part of the valuable results. It is not for the purpose of encouraging the children to carry in mind the facts of arrangement they may work out and discover. The purpose is merely to impress the point that a larger group, like nine, for example, is composed of smaller groups, like two, three, and four, for example, the purpose is to develop the idea of the column.

7. The Analysis of Groups

The preliminary exercises ought to include three of the usual steps in the study of arrangement:

1. Attention to arrangement by the pupils, when the teacher makes the arrangements.
2. Attention to arrangement by the pupils, when they make the arrangements.
3. Attention to arrangement by the pupils, when they think the arrangements.

Throughout, the oral and written language of arrangement should be used to describe the arrangements, and to direct the attention while they are being made.

Let the teacher at the outset present to the pupils a group of nine objects, let us say. First, let the objects be counted. Now show the children how a group of nine objects may be divided into three smaller groups, groups of two, three, and four.

// /// ///

Point out that nine is the same as two and three and four, and that two and three and four are the same as nine. As the group of nine is separated into the smaller groups, let the sign for each

2

3

be written, 4. Finally, ask, "Two and three and four are how

many?" Let the three groups be identical. Explain the meaning of the answer. "none," as given, write the same at the head of

3
3
the column, 4
9

Inquire if the groups of names could be divided into three different groups. Let first one child, then another, arrange the names, until the significance of the question is gained. If the children are slow in comprehending, help them. Finally, the groups of names will be divided, as:

/ III IIIII. or as III II IIIII. or as II IIIII II etc.

Let each arrangement be written, and each set of groups be numbered

1 3 2
2 2 1
3 3 3

together, and the answer written, thus 5 4 3 etc.

As the next step, let each pupil take a group of names as here, or seven, objects. Ask the question, "What three groups make ten?" Let each pupil actually make his own arrangement and write it on the board. Several arrangements of the ten objects will be made, and written.

As a final step, when the pupils have had experience in actually making the arrangements required, let them answer the questions by thinking the arrangements in their minds. (a) by looking at ten objects and thinking them arranged in groups of two, four, four, or three, five, two, etc., and writing the arrangements; (b) by thinking the largest group into the smallest groups when no objects are used.

As the pupils make and think the various arrangements, let each write his own arrangement on the board. To judge which of the arrangements will thus be written. The same can be carried out, and thus be provided.

8 Thinking the Addition

The "next step" just mentioned will be reached at the answer to the written responses of the arrangements is given. Before the pupils even will require written responses.

that will suggest and require an attack upon the problem of arranging slightly different from the one they have been

$$\begin{array}{r} 2 \\ 4 \end{array} \begin{array}{r} 1 \\ 4 \end{array}$$

$$\begin{array}{r} 2 \\ 4 \end{array} \begin{array}{r} 1 \\ 4 \end{array}$$

pursuing. Questions like these will appear: 3, 2, etc

Let the pupils now try to answer the questions. Have them do their thinking aloud, so that it will be known just what method each is using. *Do not let any pupil use the method of trying to remember the answer that was once written below the columns.* If any pupil attempts to use such a method, halt him and let him take the objects called for in the column and count them together to get his answer. Let the pupils try for a while to determine the answers by visualizing the objects, by using actual objects, by counting, but not by remembering. Many pupils will be able to secure the correct answers by roundabout methods. The purpose of such trials is to make clear to the pupils just what will be demanded of them in adding a column three digits high, to prepare them for the direct method of adding that will be shown them presently, and to require them to think the additions.

9. Demonstrating the Method

When the purposes of the preliminary exercises have been accomplished, and the pupils are ready, because of their own attempts, to give attention to the direct method of adding a column, illustrate the procedure to be followed. The addition may be illustrated as follows:

$$\begin{array}{r} 2 \quad \cdot \quad \cdot \\ 3 \quad \cdot \quad \cdot \quad \cdot \\ \hline 4 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

Point out again that such a column as the one here shown asks that two and three and four be added. Suggest that in counting the three groups of dots, one first counts two of the groups together, and next the third group. Suggest that the column asks that two and three be added and, finally, that four is to be added.

In counting the child can say easily up to ten or more
 (a) one, two, three, four, five and more, and
 (b) two and three are five, four and four are eight, five and three are five and seven are ten. If he is to be counting in words and using the third method.

Let the pupils be given very simple and easy problems such as

2

3

column as I mean (1) that two and three be added to get five, and (2) that five and four be added to get nine.

Let the pupils now attack the "word-problem" questions that are on the board. Having done this, they go slowly and carefully, and thinking and remembering as they go each step of the way in solving the problem. As the practice proceeds, speed will come, and the child will be able to solve the problems thoughtfully, and preserve his ability to solve more of them. Because the thinking work is systematically done at first, will become more and more rapid and accurate.

10 Downward Addition

In the practice exercises the pupils should be encouraged to add down the column rather than up. The combinations now illustrated have been built up with the idea that they may be added downward easily by adding downward and upward on the simple combinations that have been given already on the selection of the numbers for the combination. It is suggested to select one or the other method of adding and to use it consistently during the entire process of the general instruction. If the pupil is permitted to change without making him too method, then the work has particular emphasis for downward systematic. Let us illustrate

1

2

In the combination, 3, which will probably require the use of the

practice exercises, it is intended that the pupil will get practice on two simple combinations, namely:

One and two are three.

Three and five are eight.

Now, should the pupil be allowed to add up the column, he will miss the practice on the two combinations mentioned, which had been selected through systematic methods, and will get practice on the two following; namely:

Five and two are seven.

Seven and one are eight.

One or the other -- downward or upward addition -- must be selected, if systematic practice is to be expected.

In this discussion downward addition has been given the preference. The reasons follow:

1. The figures are written from top downward. This method of writing the figures gives a slight advantage to downward addition when the combinations are the simple ones.

2. In adding from the top down, one can set his answer down as soon as his eye and his pencil reach the bottom of the column.

3. In adding two or more columns when carrying is necessary, the figure to be carried can be written conveniently at the top of the next column, and the adding can proceed downward at once. There is no room at the bottom of the column to set down the figure that is carried.

4. Salesmen usually add the sales slip from top to bottom, and downward addition seems to be preferred by business schools in general.

5. Downward addition is slightly more accurate than upward addition.¹ However the evidence upon this point is meager and not very convincing.

¹ L. E. Cole. "Adding upward and downward," *Journal of Educational Psychology*, 3: February, 1912, pp. 83-94.

II Summary

By means of the tracing that is given in the analysis of a column, the pupils not only learn a new and different process of addition that is useful they review previous knowledge the simple addition they have learned on a new and higher level of endeavor. They review tracing in reviewing their knowledge to new processes and to new goals.

CHAPTER XIV

GROUPING INTO TENS

Argument

1. The chapter deals with the new method of grouping that pupils should learn; namely, the method that relates to the idea of ten.
2. The 36 additions whose sums extend from 11 to 18 may be learned through the application of a single method of attack.
3. Likewise, the corresponding 36 subtractions may be learned through the application of a single method of attack.
4. In each case the single method of attack relates to the idea of ten. The method in each case should be presented objectively and then practiced by the pupil.
5. Drills should follow, so that eventually the method will not be needed. The method in each case is a method of learning, not a method for later mature usage.
6. The method in each case is important as a means of preventing guesses and as an aid to understanding.
7. The new combinations, when learned, should be drilled upon as were earlier ones.
8. In the study of equal groups, at the outset a given division precedes its corresponding multiplication; later, the procedure is reversed.
9. Multiplications whose products are greater than ten do not involve in each case a piling up of smaller equal groups into a single large group, but a grouping into tens.
10. The new type of grouping is required by the system of decimal notation.
11. The method of grouping into tens should be demonstrated objectively and practiced by the pupils.
12. The method is a method of learning, not one for mature use. It is intended to aid the pupil's understanding.

In previous exercises, the pupils dealt with combinations in addition whose sums were ten or less than ten, such as
 $6 + 3$

$4 + 2$, etc. In dealing with such combinations, the pupils either brought together (or thought together) into a single group the two groups that were being dealt with in the addition. Thus, to add six and four, for example, the requirement was to discover how large a single group the two groups would make when brought together. In the exercises that will be discussed in the present chapter, the pupils deal with combinations whose sums are greater than ten, such as
 $7 + 8$

$5 + 7$, etc. In dealing with such combinations, the pupils will not be required to bring together or to think together into a single group the two groups being dealt with in the addition; they will be required to rearrange the objects in the two groups into a group of ten and so many more. Thus, to add seven and five, the requirement is to bring or to think the two groups together into a group of ten and a group of two. They will be required to write the result of

7

the regrouping as ten and two, thus: $5 + 7 = 12$. Though they may

12

designate the answer with a single word, 'twelve,' they must write it and deal with it in all other respects as 'one ten and two.'

Similarly, in previous exercises, the pupils dealt with combinations in multiplication whose products were ten or less

2×3

than ten, such as 5×3 , etc. In dealing with such combinations, the object in each case was to combine the equal groups as indicated into a single group. In the exercises of the present chapter, the pupils deal with combinations whose

5×9

products are greater than ten, such as 3×4 , etc. In dealing with such combinations, the object in each case is to recombine the equal groups, not into a single large group, but into

a group, or groups, of ten and no more, except 10 up to 100 combinations. 3 and 4, the groups are divided into groups of one ten and five, and no other tens and 5. The answers are written, 15 and 20 and so on for the others with.

Accordingly, the purpose of the exercises for the children is as much to teach the pupils a new method of remembering the objects of the groups with which they will deal as to teach certain specific combinations and relations and division, multiplication, and division. If the pupils learn only the combinations, they will be no more acquainted with the future use of the combinations than they are acquainted with the use of the combinations if they learn only the combinations. If, however, they learn a system of addition which they are learning the combinations, they will remember them with the vagaries of memory.

III Addition and Subtraction

1 The Addition Combinations

The addition combinations that precede the addition of 10 are:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

In their previous activities of addition and subtraction, the arrangement of subtraction was first observed by taking 10 away from 10, and this was followed by the study of the arrangement of addition. Thus in the study of a group of nine, for example, the group was first taken away from 10, leaving five, and then the group was added to 5, leaving 10.

five were brought together to make the original group of

4
 nine, 5. In the present activities the object is, first, to bring
 6

together two of the groups to nine, inclusive, so as to dis-

7
 cover ten and so many more, 5; next, from the addition
 12

arrangement, the corresponding arrangement of subtraction

12
 will be derived, -7 .
 -5

The work may well begin with a review of the 45 arrangements of addition that the pupils have learned. These may be set down in order as they are being reviewed. Beside these or under these the 36 combinations yet to be learned may be set down. The order in which the 36 new combinations are set down may be the order shown above. Thus, the first task may be to study the arrangements that must be learned with the groups of the nine.

2. Ten and How Many?

From the outset the pupils must be made aware of what they are undertaking to discover in the case of each arrangement to be studied. To accomplish this the exercises that have been described in Chapter XII may be quickly reviewed. The effort should be to make the pupils conscious of the special significance of the group of ten from the beginning. As an introduction to the work to be undertaken the pupils should be informed that they are to put two groups together to find out how many more than ten the two groups make; that is, to find out "ten and how many?"

Let us suppose the work begins with a study of the rearrangement of a group of nine and a group of nine. (A group of nine and a group of two would serve the purpose quite as well.) Two groups of nine are presented, thus:



or the groups may be presented like



"How many are in this group?"

"How many are in this group?"

The pupils observe the answer to each question is ten.

"We wish now to find out how many there are in all." (Or, "We wish now to find out how many there are in all.")

The question should be repeated in writing, also, as each pupil answers as before.

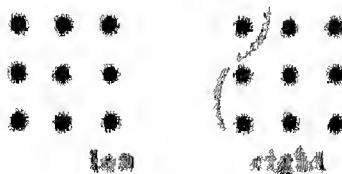
If the objects are shown in the above the question may be answered by counting the two groups as one group, and bringing them from the other to make a group of ten.

"How many are there in this group?"

"How many are in this group?"

"How many altogether?"

Or, if the objects are shown in the above they may be brought together in the groups shown, as follows:



"How many are there in all?" (Ten and eight, or eight and ten)

The answer ten and eight should now be written in the appropriate

place, 9.

18

When the demonstration has been repeated and discussed sufficiently for the pupils to understand what has been done, the corresponding subtraction arrangements may be presented.

18

eighteen is how many? -9 Through the pupils may be sufficiently familiar with the groups with which the work began so as

18

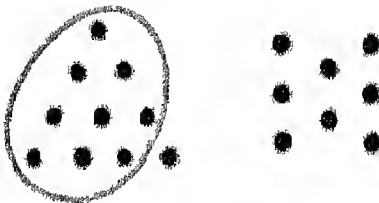
to be ready with the appropriate answer, -9 , they will profit from a

9

complete demonstration of the meaning of the arrangement.

18

The subtraction, -9 , is to be represented by a withdrawal of nine from ten and eight. Since nine cannot be withdrawn from eight, nine are withdrawn from the group of ten, and may be represented thus:



Withdrawing nine, indicated by encircling nine dots, as shown, leaves one and eight.

The steps in the study of arrangement followed in previous exercises need to be followed here also:

1. Attention to arrangement, when the teacher makes the arrangement
2. Attention to arrangement, when the pupils make the arrangement.
3. Attention to arrangement, when the pupils think the arrangement.
4. Attention to arrangement, when the objects are present only in imagination ('problem-solving,' so-called)
5. Attention to arrangement, when no objects are present

Each step should be repeated a sufficient number of times to make the pupils ready for the succeeding one. When the teacher has demonstrated the grouping of two groups into ten and so many more, the grouping may be left to the pupils to undertake in accordance with the directions given in the questions the teacher asks. Thus, when the pupils

Teaching Pupils to Think the Answer

1

8

Jane had a question like this, 8, and was not sure that she knew the answer. She did not guess. She *thought the answer*. This is the way she did it.

8 She knew that she had to make eight and six into ten and how
6 many, so she thought, "Eight and two are ten," and wrote 1
1 in ten's place at the left of the 6 column.

Jane now remembered that she took two from six to go with the eight, so she thought, "Two from six is four," and wrote 4 in one's place to the right of the 1 in ten's place in the answer.

Now, she had the answer, "Eight and six are fourteen." Jane said the answer, "Eight and six are fourteen," again and again, so that she would not forget it.

Just suppose you are not sure of the answers to these questions. Show how to *think the answers*.

9	6	5	7	6	6	9	8
8	5	9	8	7	9	9	8

2

15

Tom had a question like this, $\underline{15} - 7$, and was not sure that he knew the answer. He did not guess. He *thought the answer*. This is the way he did it.

15 Tom knew that he had ten and five, and that he had to take
 $\underline{15} - 7$ seven away. He was not sure about taking seven from fif-
8 teen all at once; so he took seven from the ten, "Seven from ten is three." He knew now that he had three left from the ten, and also five left, so he thought, "Three and five are eight," and he wrote 8 in the answer. Tom thought it all at once, something like this: "Seven from ten is three and five are eight."

Tom now knew the answer, "Seven from fifteen is eight," so he said the answer, "Seven from fifteen is eight," again and again, so that he would not forget it.

Just suppose you are not sure of the answers to these questions: (See following page.) Show how to *think the answers*.

17	11	14	12	16	18	32
8	5	2	7	8	9	1

3. Factors determining the degree of

As the possible arrangements of the atoms in the molecule of the compound and its molecular structure are determined by the nature of the atoms and the combinations in which they occur, the degree of complexity of the molecule, they should have a corresponding effect on the degree of complexity of the molecule. The molecular structure and complexity of the molecule are determined by the nature of the atoms and the combinations in which they occur. The molecular structure and complexity of the molecule are determined by the nature of the atoms and the combinations in which they occur. The molecular structure and complexity of the molecule are determined by the nature of the atoms and the combinations in which they occur.

In the construction of the molecular structure, it is necessary to be taken to have the same number of atoms in the molecule. It has been found experimentally that the degree of complexity of the molecule is determined by the nature of the atoms and the combinations in which they occur. The molecular structure and complexity of the molecule are determined by the nature of the atoms and the combinations in which they occur. The molecular structure and complexity of the molecule are determined by the nature of the atoms and the combinations in which they occur.

63	45	57	125
77	63	81	127

In the construction of the molecular structure, it is necessary to be taken to have the same number of atoms in the molecule. It has been found experimentally that the degree of complexity of the molecule is determined by the nature of the atoms and the combinations in which they occur. The molecular structure and complexity of the molecule are determined by the nature of the atoms and the combinations in which they occur. The molecular structure and complexity of the molecule are determined by the nature of the atoms and the combinations in which they occur.

example, the molecular structure of the compound is determined by the nature of the atoms and the combinations in which they occur.

and more; thus, the combination, 5, will give an answer in 'the next tens' and more, and so on. It should be borne in mind that the exercises of instruction and practice are not intended to produce a faultless mastery of the higher-decade combinations at this point. Mastery will come slowly and only as use is made of the combinations in later exercises. The purpose of the exercises is to carry the practice on the combinations to a higher level, and to help the pupils to develop an idea of relation between the simple combinations and the corresponding ones in the higher decades. It is the idea of relation that the teacher should have in mind to aid the pupils to develop.

In the exercises of column addition it is possible to organize the various possible columns of three into groups that provide a progression in difficulty from one to the other. For example, four types of columns may be provided:

A	B	C	D
2	3	6	6
3	3	8	8
4	6	5	9

These four columns may be described as follows:

Type A has an easy seen and an easy unseen combination.

Type B has an easy seen and a hard unseen combination.

Type C has a hard seen and a higher-decade combination without 'bridging' for the unseen combination.

Type D is similar to C except that bridging is involved in the unseen combination.

Such an organization may well be kept in mind for later exercises of practice. On the other hand, it may be well to

remember that the combination 9 may be no more difficult, when it is understood and mastered, than the combination 5

4, and that a higher-decade combination involving bridging

completion of the 61 combinations in multiplication and of the 81 combinations in division, it will be suggested that the method of study described in Chapter X be continued in connection with the exercises described in the preceding section of this chapter, reserving for a later grade (Grade III) the development and use of the new method.

2. Division Before Multiplication

As was indicated in the section on the study of equal groups in Chapter X, when the fact of multiplication is derived from the study of a group, it comes as the answer to a question of division. The object of the study at the time is a given group of objects. In order to call further attention to the group, the teacher inquires about the number of equal groups of a given size in the group being studied. The children discover the answer by dividing the group into the equal groups indicated in the question.

In the study of groups to ten, inclusive, the following division questions were asked and answered:

2/1, 2/2, 3/3, 2/4, 4/4, 1/5, 2/5, 5/5

In the studies of arrangement outlined in the preceding section of this chapter, certain divisions and corresponding multiplications are more or less obvious. For example, in answering the addition question, "How many are nine and nine?" the two groups of nine are before the eyes of the pupils for study:

• • •	• • •
• • •	• • •
• • •	• • •

When the two groups have been combined into ten and eight — eighteen — the question, "How many nines are in eighteen?" $9/18$, is not difficult to answer. Indeed, the answer is before the children's eyes from the beginning, and they need only to have the answer called to their attention by the proper question.

The answers that are thus discovered to the division questions are the answers needed for the corresponding questions when put in the form of multiplications. Thus, the question

$$\begin{array}{r} 6 \\ 3 \overline{)18} \end{array}$$

and answer, 3, provide the answer to the question, "Six

$$\begin{array}{r} 3 \\ \times 6 \\ \hline \end{array}$$

threes are how many?"

The five steps in the study of arrangement will need to be taken one by one in connection with the 15 new division and the 15 new multiplication combinations.

V. A NEW STUDY OF GROUPING

In the exercises with equal groups that have been referred to up to this point, division has preceded multiplication. In the new study of grouping, which is begun after a considerable time has elapsed, the pupils undertake to put together the groups to nine, inclusive, to discover in each case "how many tens?"

1. Making a Table of Combinations

The work may be begun with a review of the combinations already learned. These may be set down with the ones yet to be learned in 'tables.' The tables will provide a useful means, at first of setting down the answer in each combination as it is discovered, and finally of conducting reviews and drills. When the tables are first presented, the answers to the combinations yet to be learned may be omitted:

1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{8}$	$\overline{9}$
2	2	2	2	2	2	2	2	2
1	2	3	4	5	6	7	8	9
$\overline{2}$	$\overline{4}$	$\overline{6}$	$\overline{8}$	$\overline{10}$	$\overline{12}$	$\overline{14}$	$\overline{16}$	$\overline{18}$

Teaching to Find How Many Tens

Multiplication is a way of finding out exactly. Usually, most of multiplication is a way of finding out exactly how many tens, since every number that is more than 9 and up to 100 is written by putting a figure in ten's place. There is no separate figure for eleven; we write it 1 ten and 1, then, 11. There is no separate figure for twenty-five, or forty-seven, or sixty-nine; we just use certain figures from 1 to 9 in different places, thus, 25, 47, 69.

$$\begin{array}{r} 2 \ 4 \ 3 \\ 5 \ 6 \ 9 \end{array}$$

Of course, combinations like $\begin{array}{r} 2 \ 2 \ 3 \\ 5 \ 6 \ 9 \end{array}$, which have answers less

than ten, do not mean finding out how many tens. But combina-

$$\begin{array}{r} 5 \ 2 \ 4 \ 3 \ 2 \\ 5 \ 6 \ 9 \end{array}$$

tions like $\begin{array}{r} 5 \ 2 \ 4 \ 3 \ 2 \\ 5 \ 6 \ 9 \end{array}$, which have answers larger than 9, mean finding out how many tens. Suppose we want to find out how many are four tens, or five threes. We try to find out how many

$$\begin{array}{r} 4 \ 3 \\ 10 \ 15 \end{array}$$

tens, then, 4, 3, and we learn that four tens are 1 ten and six (we say sixteen), and that five threes are 1 ten and five (we say fifteen).

All the combinations you will now have to learn are combina-

$$\begin{array}{r} 3 \\ 6 \end{array}$$

tions which ask you to find out how many tens. Thus, 7 means

$$\begin{array}{r} 7 \\ 6 \end{array}$$

"seven threes are how many tens?" 4 means "four sixes are how

$$\begin{array}{r} 4 \\ 9 \end{array}$$

many tens?" 7 means "seven nines are how many tens?" The rest of the combinations ask the same kind of question. How many tens?

Let us see if you can find out how many tens there are in some of the combinations which you do not already know. Let us start with the 5's. Perhaps you can work most of them out for yourself.

The work is begun using groups of five for the obvious reason that they group readily into tens. Note the reason for following with groups of nine, page 308

The 5's in Multiplication and Division

We may show one five, thus $\begin{array}{c} * \\ * \end{array}$ By including all the groups of dots, or by counting them, we can tell that one five is $\frac{5}{5}$

We may show two fives, thus $\begin{array}{c} * * \\ * * \end{array}$ How many are there just, that is, how many times? By including we can tell that two fives are 1 ten. We say "Two fives are ten" and write 2, $\frac{10}{5}$

We may show three fives, thus $\begin{array}{c} * * * \\ * * * \end{array}$ Three fives are how many times? We can see that 15 is three fives, and that there are five groups in ten fives, that three fives are 5 tens, and 15. We say, "Three fives are fifteen," and write 3, $\frac{15}{5}$

Now, let us work the three you do not already know. How many tens. Let us show four fives



The first two fives go together to make 1 ten, and the second two fives go together to make 1 ten. It is easy to record this. There is the way we find out that four fives are 2 tens. We say, "Four fives are twenty," and write 2, $\frac{20}{5}$

You should now turn back to the table of 5's and write the answer to the 4 multiplication. Also write the answers to the 4 division in the table of 5's. If four fives are twenty, how many are five fives?

Do you remember the kind of question a sign like this $\frac{20}{5}$ asks? It asks, "How many fives are twenty?" (over the answer) (20) Give the answer.

3

3 asks, *how many tens?* Let us show *five tens*, and then count the tens.



5

3

Five tens are ~~thirty~~ *—?* 3? Write the answer in the 3 combination in the table of 5's.

5)25? Give the answer.

How many are *six tens*, that is, *how many tens?* Let us show *six tens*, and count the tens.



5

6

Six tens are ~~thirty~~ *—?* 6? Write the answer in the 6 combination in the table of 5's. See whether you can now write the answer to

6

the 5 combination in the table of 6's.

5)30? Give the answer.

6)30? Give the answer.

Seven tens are how many tens? Count them.



5

Seven tens are ~~—~~ *—?* 7? Write the answer in the table of 5's.

7

Five sevens are ~~—~~ *—?* 5? Write the answer in the table of 7's.

5)35? Give the answer.

7)35? Give the answer.

Eight tens are here counting from 1 to 80.



Eight tens are ... the first ten ... the second ten ... the third ten ... the fourth ten ... the fifth ten ... the sixth ten ... the seventh ten ... the eighth ten

Five eights are ... the first eight ... the second eight ... the third eight ... the fourth eight ... the fifth eight

540? Give the answer

540? Give the answer

Nine tens are here counting from 1 to 90.



Nine tens are ... the first ten ... the second ten ... the third ten ... the fourth ten ... the fifth ten ... the sixth ten ... the seventh ten ... the eighth ten ... the ninth ten

Five eights are ... the first eight ... the second eight ... the third eight ... the fourth eight ... the fifth eight

545? Give the answer

545? Give the answer

Remember with the 10

Say the answers to these ... the column, then up the next column

5	5	5	5	5	5	5	5	5
1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9
5	5	5	5	5	5	5	5	5

575 575 575 575 575 575 575 575 575

115 210 315 420 525 630 735 840 945

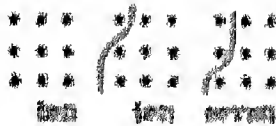
Say the answers ... forget them

Teaching the 9's in Multiplication and Division

You have just found out that it is not hard to work the answers to the combinations in the table of 3's, that is, to find out how many tens. We will now work out the answers to the combinations in the table of 9's, and you will see that they are not very hard either, since the number nine is right next to the number ten.

Since you already know these two combinations, 1. and 2., we will start with three nines, 3.

Three nines are how many tens? Count the tens.



Three nines are $\frac{9}{3} = 3$? Write the answer in the table of 9's.

Nine threes are $\frac{9}{3} = 3$? Write the answer in the table of 3's.

$9 \overline{)27}$? Give the answer.

$3 \overline{)27}$? Give the answer.

Four nines are how many tens? Count the tens.



Four nines are $\frac{9}{4} = 4$? Write the answer in the table of 9's.

Nine fours are $\frac{9}{4} = 9$? Write the answer in the table of 4's.

$9 \overline{)36}$? Give the answer.

$4 \overline{)36}$? Give the answer.

Five nines are how many tens? Look at your table of 9's. Do you remember how you found out 5?

See if you get the same answer as the other 100



100? Give the answer

100? Give the answer

See if you get the same answer as the other 100



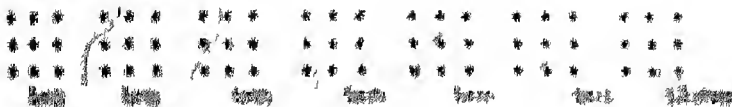
See if you get the same answer as the other 100

See if you get the same answer as the other 100

100? Give the answer

100? Give the answer

See if you get the same answer as the other 100



See if you get the same answer as the other 100

See if you get the same answer as the other 100

100? Give the answer

100? Give the answer

See if you get the same answer as the other 100



See if you get the same answer as the other 100

8

Nine eights are -----? 9? Write the answer in the table of V_1 .

9)72? Give the answer

8)72? Give the answer

Nine nines are how many tens? Count the tens.



9

Nine nines are -----? 9? Write the answer in the table of V_1 .

9)81? Give the answer.

Examples with the V_1

Say the answers to these examples across the rows, then down the columns, then up the columns.

9	9	9	9	9	9	9	9	9
<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>	<u>6</u>	<u>7</u>	<u>8</u>	<u>9</u>
1	2	3	4	5	6	7	8	9
9	9	9	9	9	9	9	9	9

9)9	9)18	9)27	9)36	9)45	9)54	9)63	9)72	9)81
1)9	2)18	3)27	4)36	5)45	6)54	7)63	8)72	9)81

Say the answers over and over until you are sure you will not forget them. Turn back to page 307 and say the answers to these examples.

Exercises similar to the ones just indicated should be developed around the following topics

- Teaching the 8's in Multiplication and Division*
- Teaching the 6's in Multiplication and Division*
- Teaching the 7's in Multiplication and Division*

3. The Multiplication and Division Table

A useful device for reviews, drills, and for the solution of examples in multiplication and division that immediately follow the learning of the simple combinations is the multiplication and division table shown on the next page.

卷之四十五

[illegible][illegible][illegible]

Suppose that we have a function $f(x)$ which is continuous on the interval $[a, b]$. Then the function $f(x)$ is integrable on $[a, b]$ and the integral $\int_a^b f(x) dx$ exists. If $f(x)$ is continuous on $[a, b]$, then $f(x)$ is integrable on $[a, b]$ and the integral $\int_a^b f(x) dx$ exists. If $f(x)$ is continuous on $[a, b]$, then $f(x)$ is integrable on $[a, b]$ and the integral $\int_a^b f(x) dx$ exists.

Copy this table and send it to the publisher and printer of the paper in which you already have an advertisement. Give the table to the publisher and printer of the paper in which you already have an advertisement. Give the table to the publisher and printer of the paper in which you already have an advertisement.

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 卷之四
 四庫全書

A useful course of instruction is to consider them as complexes of acts in the structure of the mind. There are not elements, but complexes, imposed and imposed. Especially true with acts of which acts are acts.

are inclined to consider as the difficult ones. The products that come in the several decades from the 20's on are these:

20,	21,	22,	23,	27,	28
30,	32,	33,	36		
40,	42,	43,	46,	49	
54,	56				
63,	64				
72					
81					

In using these products in review, a product can be given, and the pupils can give the combination, or combinations, that produce it. Thus, when 63 is given, the pupils can give the combinations, "Seven nines are sixty-three," and

9 7

"Nine sevens are sixty-three," or write them, 7 and 9.

63 63

CHAPTER 1

THE PURPOSE OF THE STUDY

1. The purpose of the study

1. The idea of law, which suggests the possibility of a new method of action, may develop into a new method of action. If so, it may serve to clarify the nature of the government of the world.

2. The paper should have been written in a way that the reader can see the author's own method of action.

3. Using the same idea and method of action, it is possible to find the value of the government of the world. It is possible, it may be said.

(a) There are no conditions, and no conditions are possible.

(b) There may be conditions, and no conditions are possible.

(c) There may be conditions, and no conditions are possible.

(d) There are conditions, and no conditions are possible.

(e) There may be conditions, and no conditions are possible.

4. Each application of the idea is a new method of action.

5. The development of the idea is a new method of action. It is the paper's development of the idea of law, which suggests the method of action.

6. The long-term idea is a new method of action. It is the idea of law, which suggests the method of action. The idea of law is a new method of action. It is the idea of law, which suggests the method of action.

2. The purpose of the study

The purpose of the present study is to show the method of action by which the paper is written. It is the method of action by which the paper is written. It is the method of action by which the paper is written.

position which goes with it, in developing an understanding of the complex processes of the least fundamental operations.

The pupil's manner is later must move ahead from the simpler processes to the complex processes. He may learn the latter in either of the following ways. (1) He may learn them merely as operations to be performed without understanding what he is attempting to do in each case or why he should do it. (2) He may learn them as operations to be performed, developing in connection with each such understanding as may come from the teacher's explanations. (3) He may learn them as operations that are held together by the common bond of an idea that is his own personal possession and that he may explain for himself each in terms of this central idea. When the pupil learns the complex processes by this last method, he may learn as he proceeds to rely more and more upon his own ability to explain. Since the idea develops in clarity and strength as he proceeds, he is in position to discover that each apparently new process to be learned is in reality not so new, and to learn as he attempts each apparently more difficult process that in reality it is less difficult than preceding ones, because he is more able to provide his own explanation for it than he was for those that preceded.

11. WHAT THE PUPIL HAS LEARNED ABOUT TENS

If the pupil has been through such exercises as have been described in the three chapters immediately preceding, he has learned to pay particular attention to the idea of ten and to give it a special significance. He has learned, in some degree at least, to deal with tens just as he deals with units, and to translate chance groups into tens. Having been required to use the idea of ten more than any other number idea, he has developed for it a special acquaintance. If the teacher now will so direct, he can with ease continue to attach to the idea of ten a special significance, and to look to this idea to suggest the clue to the answers to ques-

20

In the written form, 2. the pupil is to realize that he is to multiply tens, but is to proceed just as he does in multiplying units. He thinks, when he multiplies, "Three twos are six," but realizes that his answer, "six," means six tens, and must be written in ten's position. In order to place the 6 in its proper position, one writes a zero at the outset of the proceeding, so that, when the 6 is written, it

30

will show six tens: 3. Note that in this case, as in every

60

other when the zero is involved, the zero is used to hold position: one does not multiply the zero; he merely uses a zero because he knows, ahead of time, that his answer will be tens, and he wants to write the answer in ten's position.

34

In multiplying two thirty-fours, 2. one thinks, "Two fours are eight," and writes 8 in unit's place. Next, he thinks, "Two threes are six" and writes the 6 under the 3

34

in ten's place: 2.

68

In similar manner, the pupils may learn how to multiply hundreds and thousands 'just like units.' When they take care to write each part of their answer in its proper position, position keeps the answers clear and relieves them from any thinking other than to proceed in each case to multiply 'just like units.'

IV. CARRYING TENS IN ADDITION AND MULTIPLICATION

The following paragraphs are included to illustrate both the procedure that may be followed in introducing pupils to the idea of carrying tens and the emphasis that may be placed upon the idea of ten that the pupils may now have partly formed in their minds. It is to be understood that each presentation should be followed with enough practice

to fix the procedure in mind. It is to be understood further that the notion of carrying, or the meaning of carrying, needs to be developed in the minds of the pupils before they are directed to begin the necessary exercises of practice.

Teaching the Carrying of Tens in Addition

You have already learned a good many things about tens. You know how to write them and read them, and how to add, subtract, and multiply them. There is something more for you to learn about tens, and that is how to carry them. Let us see how tens are carried in addition.

Suppose we wish to add 25 and 43. We first add the units, "five and eight are thirteen." Now 13, as you remember, is 1 ten and 3. Since we cannot write all the 1 ten and 3 in unit's place in the answer, we write as much of it as we can, that is, we write the 3. But we still have the 1 ten. The place to put the 1 ten is in the next column with the 2 tens and the 4 tens. We do not have to write it there, but can just think of there with the 2 tens and the 4 tens. That makes 6 tens, and 2 tens and 4 tens to be added. We think, "One and two are three, and three are seven," and write 7, (which is 7 tens), in its proper place. This is the way we think as we add. "Five and eight are thirteen." Think of "One and two are three and four are seven." (Write 7.)

Sometimes in order to help us to remember about the 1 (ten) to carry," we write a little 1 (which we have seen put above the figure at the top of the next column. But we never do that unless we have to. You can do that, if you wish. But see if you are not good enough at remembering without having to write the number you have to carry.

Teaching the Carrying of Tens in Multiplication

Tens are carried in multiplication just as they are carried in addition. Suppose we wish to multiply 25 by 2. We first multiply the units, and then we multiply the tens. We think, "Two sixes are twelve." Now 12, as you remember, is 1 ten and 2. Since we cannot write all the 1 ten and 2 in unit's place in the answer, we write as much of it as we can, that is, we write the 2, and remember that we have 1 (ten) to carry in the ten's place in the next part of our answer. We now multiply the 2 tens by 2 tens.

threes are six." We know that there is 1 ten, and we remember that we have 1 ten to take care of which we did not write. So we think like this, "Two threes are six, and one are seven," and we write 7 in the proper place.

Carrying Tens in Addition and Multiplication

We carry from tens to hundreds just as we carry from units to tens. You already know how to carry tens in an example like this one. We think, "Six and eight are fourteen," write 4, and carry 1 (ten): "One and two are three and four are seven," and write 7. The answer is 74.

$$\begin{array}{r} 26 \\ 48 \\ \hline 74 \end{array}$$

Now let us carry it in to carry in an example like this, and how much it is like the carrying in the example just above.

$$\begin{array}{r} 200 \\ 480 \\ \hline 740 \end{array}$$

We have no units to add, so we write 0 in order to be able to write the answer we get in its proper place. Next we add the tens just as if they were units. "Six and eight are fourteen." The answer is, of course, 14 tens; that is, 10 tens and 4 tens, and it belongs in ten's place, but since we cannot write all of it in ten's place, we write the 4 in ten's place, and carry the 10 tens (which is the same as 1 hundred), to the next column. We now have 1 hundred, and 2 hundred, and 4 hundred to add. We add, "One and two are three and four are seven," and write 7 in hundred's place. The answer is 740. We think, "Nothing," and write 0, in order to put the answer we get in its proper place. We think, "Six and eight are fourteen," and write 4, and carry 1. We think, "One and two are three, and four are seven," and write 7. We think, "Three and two are five," and write 5. We think, "Six and eight are fourteen," and write 4, and carry 1. We think, "One and two are three, and four are seven," and write 7. The answer is 745.

$$\begin{array}{r} 200 \\ 480 \\ \hline 745 \end{array}$$

You remember how to carry from units to tens in multiplication.

If the example in multiplication is like this, we think as we multiply: "Two sixes are twelve," and we write 2 and remember that we have 1 ten to carry. Now we think, "Two threes are six, and one are seven," and we write 7. The answer is 72.

$$\begin{array}{r} 36 \\ 2 \\ \hline \end{array}$$

See if you can tell how to carry in multiplication in an example like this one,

$$\begin{array}{r} 360 \\ 2 \\ \hline \end{array}$$

There are no units to multiply. We are now 0 in units & tens.
Next we multiply the tens. "Two tens are twenty." (Write 0 in
12 tens, that is, 10 and 2 tens, we write the 2 in tens & place
remember to carry the 10 tens, which is 1 hundred. Then, we
multiply the hundreds. "Two tens are six, and two are seven."
We write 7 in its proper place. The answer is 720.

When we multiply, we do not need to be thinking about
units, tens, and hundreds any more than just as we
member to write each answer in its proper place and to carry
when we need to.

We notice the 0, think "Nothing," and write 0.

We think, "Two tens are twenty," write 2 and remember to
carry 1.

We think, "Two tens are six, and two are seven," and write 7.

In this last example, we think:

$$\begin{array}{r} 364 \\ \times 2 \\ \hline 728 \end{array}$$

"Two fours are eight." (Write 8.)

"Two sixes are twelve." (Write 2, & 1 in carry.)

"Two threes are six, and two are seven." (Write 7.) The an-
swer is 728.

V. Using Tens in Multiplication

The following paragraphs are included for guidance to the
the procedure that may be followed in introducing pupils to
the idea of using tens in multiplication and the emphasis here
may be placed upon the idea of tens that the pupils may have
have partly formed in their minds. To the pupils who have
never had his attention called to the question of using 10
group of ten and who has not already brought to mind the
the idea in his work up to this point the statement may
appear obscure and not so helpful. To the pupils who have
been developing his idea of tens and making use of tens
in thinking about the process he has just learned, at this
point, the introduction will be easy and successful. In fact,
the latter pupil will be able to meet the requirements of the

trained halfway, and to contribute a good deal from his previous experience toward making part of the explanations himself.

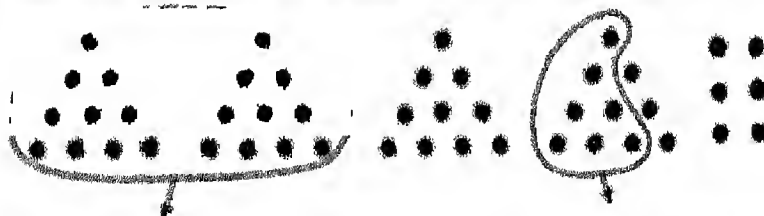
1. Illustrative Examples

Teaching the Use of Ten in Subtraction

In subtracting, we sometimes have to take away more ones than we have ones to start with. When we do, we just 'carry back' and use one of the tens we have, and then take away ones from the ten and more.

Notice how we subtract 27 from 40.

40 To start with, we have 40, that is, *four tens and six*. Let us
27 show *four tens and six dots*, and then pretend to take away
two tens and seven by putting circles around that many dots.



We take away two tens.

We take away seven.

One ten and nine are left.

Since we do not have enough ones to take away seven, we carry back and use one of the four tens with the six; then we take away seven from the ten and six.

$$\begin{array}{r} 40 \\ 27 \\ \hline 19 \end{array}$$

We think, "Seven from sixteen is nine," and write 9.

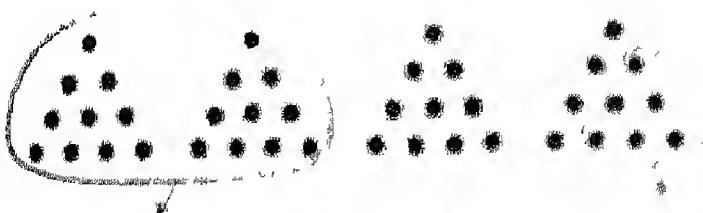
Next, we take away two tens. We remember that we have used one of the four tens already and that we must take two tens from three tens.

We think, "Two from three is one," and write 1 in *ten's place*.

In subtracting, we sometimes have to take away ones when we have no ones to start with. When we do, we just carry back and use one of the tens we have; then take away ones from the ten.

Notice now how we subtract 26 from 40.

40 To start with, we have 40, that is *four tens*. Let us show *four*
26 *tens dots*, and then pretend to take away *two tens and six* by
putting circles around that many dots.



We take away two tens. We take away one
(one ten and four are left)

Since we have no ones to take away yet, we carry back and use one of the four tens. Then we take away six from the ten that we carry back.

We think, "Six from ten is four," and write 4.

Next, we take away two tens. We remember that we have carried back and used one of the four tens, and that we must subtract two tens from three tens.

We think, "Two from three is one," and write 3 in the tens place.

Carrying Fractions with Tens

In subtracting, we sometimes carry back one unit of tens, or hundreds, or thousands, when we have one unit of tens, or hundreds, or thousands to take away. If we have one unit of tens, or hundreds, or thousands to take away, we write down what we have when we have nothing to take away.

853 We think, "Seven from thirteen." We cannot take 7 from 3 tens, so we have only four tens, and we take 7 from 4 tens, leaving 1 ten away from the four tens, so we write back 1 ten more. We write 1. We now think, "Four from eight."

8534 We think, "One from four." There are four tens left.
407 "Four"; and "Four from eight."
4463

In subtracting, we sometimes have to remember how many hundreds we have at the start. It is not very difficult to remember.

9603 We think, "Six from thirteen." We have 9 tens at the start.
2476 Did we have a ten to carry? Yes. We have 10 tens at the start, and when we carry back one ten, we have 11 tens. We now have 10 tens. If we carry back one ten, we have 11 tens.

Just think, "Seven from nine." "Seven from nine is two." "Nine minus seven is two."

We think 'Three from five.' We had to use one of our tens.
 And we have a ten to use." Then "Notice that we had 600
 there at the start, and when we carry back and use one of
 those, we were left 500 there. It is easy to take 400 from
 500. Just think, 'Five from nine', 'Ten from nine', and
 'Four from five'."

The 'carrying back' and 'using' of hundreds, thousands,
 etc., as presented in a manner similar to the presentation of
 the carrying back and using of tens. Since the pupil has
 learned that tens, hundreds, etc., are subtracted just like ones,
 concrete dramatization of the carrying back and using of
 hundreds and thousands is unnecessary.

2. Decomposition versus Equal Additions

In the illustrations exercises that have been included it
 will be seen that the method of decreasing in the minuend,
 known as the 'method of decomposition,' has been empha-
 sized in preference to the method of increasing in the subtra-
 hend, known as the 'method of equal additions.' Although
 certain experimental evidence appears to indicate a slight
 superiority in accuracy and speed of the latter method over
 the former, the former method has been chosen for empha-
 sis, because it is consistent with the ideas and methods of
 thinking relative to the decimal notation and the place
 value of the numerals that the pupils up to this point have
 been acquiring. Slighter ease in operation in later work in
 arithmetic does not justify a choice. Neither is a choice
 justified by personal preference or by adult rationalization,

F. B. Hallard. "Norms of performance in the fundamental
 processes of arithmetic, with suggestions for their improvement."
Journal of Experimental Pedagogy, 3: March 3, 1913, pp. 9-20, esp. p. 9.

W. W. McClelland. "The different methods
 of subtraction." *Journal of Experimental Pedagogy*, 4: December 5,
 1913, 293-299, esp. p. 298.

W. H. Wirth. "'Equal additions' versus 'decomposition' in teach-
 ing subtraction: An experimental research." Pp. 220, 270. *Journal of
 Experimental Pedagogy*, 5: June 5 and December 6, 1920, pp. 207-220,
 261-270.

as commonly illustrated by the interest already for a definite 'borrowing' through the subtraction of the tens from the hundreds and cents for the children but we can also add tens, hundreds and tens, and so on, and by the repetition of equal additions in the following examples.

The method of equal additions is the method of borrowing, and though not so obviously 'material', as the other method, it is plain. It rests, of course, on the notion that if a number is added to unequal their difference is unaffected. If the difference between my height and Tony's is one foot, say, then if we both add the same to our height, the difference between us is still the same. If I am 50 and Tony is 11 the difference between us is 39 and if I am 80 and Tony is 41 the difference between us is still 39 when each of us has added 30 to our height. Is this other twenty years later?

The method of decomposition, or the method of 'borrowing tens', that is, of making one of the tens a hundred, is the method. This the pupil has learned in the addition of tens.

In the subtraction, the method of borrowing is used, and when the minuend is a hundred and a ten and a five, the original subtraction is changed directly from the ten. No ten has to be borrowed, the idea of 'paying back' and for the ten brought in from the hundred (which is a very common explanation) is added to the hundred and the hundred is subtracted. At the end, the hundred is the hundred immediately preceding the child's hundred for the subtraction. It is not a difficult thing to do, and it is the method of decomposition. At the beginning of this section, the child learns the method of decomposition with all the necessary explanation. The subtraction is to be made directly.

¹ E. A. Greening Lamborn *Lesson in Mathematics* (University Press, London, 1930) pp. 21-22

only new thing to be learned is a new application in a slightly modified, though slightly different, situation.

Moreover, since the pupil has learned to subtract tens, hundreds, etc., just like ones, he can quickly make the step to 'carrying back' and 'losing' a hundred, or a thousand, when necessary, just as he carries back and loses a ten.

But ~~disadvantages~~ offer no objection to the procedure of teaching the pupil the other method after he has learned the one - provided in each case the learning is raised to the level of understanding. Understanding a second method and its contrasts with the first may, indeed, aid in furthering the pupil's understanding of the first; but care must then be exercised that the pupil is led to make distinctions and to see them clearly, otherwise, the teaching of a second method may lead to confusion.

VI INTENSIVE TENS

The following paragraphs are included to illustrate both the procedure that may be followed in introducing pupils to the idea of dividing tens and the emphasis that may be placed upon the idea, already considerably developed, that tens are dealt with in all operations 'just like units'. As in the illustrative exercises that have preceded, it is to be understood that each step of the presentation should be followed with sufficient practice to fix it in mind before proceeding to the next step.

1. Illustrative Examples

Teaching the Dividing of Tens

How do you think *tens* are divided?

Do you remember that *tens* are added, subtracted, and multiplied just the way *units* are added, subtracted, and multiplied; that *tens* are carried just the way *units* are carried; — in fact, that you can do everything with *tens* that you can do with *units* so long as you remember that they are *tens*? How are *tens* divided? You can guess the answer, it is so easy. *Tens* are divided exactly the

way units are divided. All your work for this is to remember that there are tens, and to write the tens a hundred or more or fewer.

Suppose we want to find out how many hundreds there are in 230.

230

We can see that 230 is 23 tens and 0 units. We think, "There are 23 tens." (Of course, we remember that it really is 23 tens, though we do not have to say "tens" at all. We write the 2 above the 3, because we want the 2 (tens) in the same position as the 3 (tens). We now think, "Two hundreds are 200, and write 6 under 6, and draw a line. Then we add 30 to 200, and write 3 under the line, and think, "There are 23 tens." We write the 3 above the 0, because the 3 is 3 tens, and we want it to have the same position as the 3 tens. We now think, "There are 23 tens," and write 2 under 2, and draw a line. Our answer is 23, which tells us that there are 23 hundreds in 230. This is the way our example looks when we have it finished.

23

230

0

0

0

81 Suppose we want to find out how many hundreds there are in 7567. We see that the first number in the thousands is 7 (seven), and we know that there are 10 hundreds in 1000.

56 we look at the 56 (tens) and add 10 hundreds to 1000.

7 we look at the 7 (tens) and add 10 hundreds to 1000.

6 we look at the 6 (tens) and add 10 hundreds to 1000.

7 we look at the 7 (tens) and add 10 hundreds to 1000.

6 we look at the 6 (tens) and add 10 hundreds to 1000.

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7 we look at the 7 (tens) and add 10 hundreds to 1000.

6 we look at the 6 (tens) and add 10 hundreds to 1000.

Teaching Trial Division

In trial division we try the answer we think is correct to see whether it is correct. If the answer is not correct, we try another one. We then have to try answers to see whether they are correct.

Suppose we are trying to find how many twos in nine. We think, "Twos in nine, four," and write 4. We think, "Four twos are eight," and write 8 under 9, subtract, and write 1. Since the 1 we have is less than 2, we know that 4 is the right trial answer.

Since the 2 we get after subtracting is less than the 2, we know that 4 is the right trial answer.

Suppose we are not careful, and happen to try 5 as an answer in this example. We think, "Twos in nine, five," and write 5. We think, "Five twos are ten," and write 10. But we can see that 10 is more than 9, so we know that 5 is not the right trial answer, and we must erase and try again.

Suppose we are not careful, and happen to try 3 as an answer in this example. We think, "Twos in nine, three," and write 3. We think, "Three twos are six," write 6, and subtract. The 3 we get by subtracting is more than 2, so we know that the trial answer is not the correct one. Then we must erase, and try again.

Suppose we had made the mistake in this example of trying 4 as an answer. We multiply, and subtract. Since the 4 we get by subtracting is more than the 2, we know that our trial answer is not the correct one.

Of course, the thing to do is to think carefully, so as not to make such mistakes; then our first trial answer will be the correct trial answer.

Teaching the Use of Trial Division in Dividing Tens

Suppose we want to find how many twos in 98. We see that we first have 9 tens to divide, and we know that tens are divided just like units; so we think, "Twos in nine, four," and we write 4 in our answer over the 9. We know that since the 9 is 9 tens the 4 in our answer must be 4 tens; so we write it in the same place that the 9 has. Next we multi-

but just write a 0 in our answer beside the 3, so that it will show 3 tens. (That answer is 30, which tells us that there are 30 twos in 60.)

Suppose we want to find how many fours there are in 60. We first pay attention to the 6 (tens). We think, "Fours in six, one," and write 1 over the 6 so it will be in ten's place. We multiply, "One four is four," and write 4 under 6, subtract, and write 2. The 2 shows that we still have 2 tens yet to be divided, and as we have no units to be divided, we write 0 beside the 2, which now shows 20. We think, "Fours in twenty, five," write 5 above 0 in the answer, and multiply. We think, "Five fours are twenty," and write 20 under 20. Our answer is 15, which tells us that there are 15 fours in 60.

Teaching the Dividing of Hundreds and Tens

How would you think *hundreds* are divided? Perhaps you remember that *hundreds* are added, subtracted, and multiplied just as *tens* and *units* are added, subtracted, and multiplied; that *hundreds* are carried just as *tens* are carried; in fact, that you can do everything with *hundreds* that you can do with *tens* and *units*, so long as you remember that they are *hundreds*. How are *hundreds* divided? The answer is easy. *Hundreds* are divided just as *tens* and *units* are divided. All you need to do is remember that they are *hundreds*.

Suppose we want to find how many threes there are in 690. First we notice the 6 (hundreds). We think, "Threes in six, two," and we write 2 in the answer just over the 6, because we know that the 2 is really 2 (hundreds). We multiply, "Two threes are six," and write 6 under 6. Since 6 is the same as 6, we know that we have divided all of our *hundreds*. We now have 9 *tens* to divide. We think, "Threes in nine, three," and we write 3 in our answer over the 9, because we know that the 3 is really 3 (tens). We multiply, "Three threes are nine," and we write 9 under 9. We can see that we have divided all our *tens*, and we have 0 *units* to divide; so we write 0. Since there is 0, or nothing, to divide, we just write 0 in our answer, to show that the 3 is 3 *tens* and the 2 is 2 *hundreds*. Our answer is 230, which tells us that there are 230 threes in 690.

Let us see how we divide *hundreds* and *tens* when we have to use trial division.

Suppose we want to find how many times there are in 2500. We notice the 2, and can tell that there are two times as 2 as the first at 25, which we can see is really 25 hundred. We have found how many times there are in 25 hundred. We think, "Five is twenty-five, five, and therefore we know that five five really means five hundred, we write 5 in the answer just over the 5 in order to make it show 5 hundred. We now multiply, "Five times two hundred, five times one hundred, 25 under 25, subtract and write 0. Then 5 times 1000 we still have 1 hundred yet to divide, and we can see that we have 5 tens yet to divide, so we write 5 under the 5. We have now 15 (tens) to divide. We think, "Five is fifteen, fifteen," and we write 3 in our answer over the 5, because the 3 is really 3000. We multiply, "Three times one hundred," and we write 3 hundred 0. We can see that we have divided all our tens, and we have 0, so no, units yet to divide. Since we have nothing to divide, we do not think about dividing, but write a 0 in our answer over the 0, to hold the unit's place, and we reach the 5 above 5 hundred and the 5 show 5 hundred. Our answer is 500, which, like the first division, was 530 fives in 2650.

2 The Forms of Division

With the presentation of the present form of the short division the teacher is required to teach a somewhat complex method which has to be grouped together roughly as the forms of division. Consideration must be given to each, separately as (a) 'short' and 'long' division, (b) remainder and this form, and (c) remainders in division.

a. *Short and Long Division* With regard to the short and long division, our suggestions are that the pupil learn the one called 'long form' from the beginning and that the difference between the two forms be noted in the early grades. At the earliest when the pupil is asked, "How many twos are in eight?" usually he answers "four" and in writing thus, 2/8, he should learn to give the complete answer, "Four twos are eight," and he should learn to write the complete answer as completely as the present form of writing divisions will permit. Thus, as the present form of

teachers have recommended, when the pupil writes the an-

swer to the question, $2\overline{)8}$, he writes it as indicated: $2\overline{)8}$.

Learning to write the 'complete' answer from the beginning, the pupil, by giving a multiplication answer to his division question, learns to multiply as part of his procedure in division, without thinking of the multiplying as something extra to do. Having multiplied in the giving of answers to such questions as $2\overline{)8}$, $7\overline{)63}$, $6\overline{)24}$, etc., without thinking of the multiplications other than as the answers that the division questions have required of him, he is ready to take up his dividing of tens and hundreds in each of which two or more division questions are asked that require two or more multiplication answers. In other words, the need to multiply when dividing does not then suddenly arise as something new to be done in a new kind of division; multiplying is part of the total procedure from the outset, and since it is, such extra attention as is required in the more complex processes may be given to the procedure of subtraction. To put the matter in another way, when the so-called 'long form' is learned from the beginning, attention, not being required for the learning of a 'new' form in the later processes, can be given almost exclusively to the new ideas of procedure that must be developed in connection with them.

Moreover, 'short' division is in reality a 'short-cut' process, which, because of its difficulty, should be learned only when the 'long' step-by-step process has become thoroughly understood. It may be employed with one-digit divisors only and has, therefore, only a limited use. Such facts as these help to explain the mistaken psychology of the teacher who presents short division first, thinking it is the easier. Short division may be used only in the easier divisions (those with one-digit divisors); long division *must* be used in the more difficult divisions (those with two-or-more-digit divisors). Therefore, reasons many a teacher, short division

is easy, and long division is difficult. We can see how we should consider it that short cuts are always a very difficult for beginners and that the more complicated the work, the more shown from the outset. The reason is no doubt the beginning of grasp. The long division form is really the reason for this because it shows more and requires less explanation in the first place in his mind while he is working, it is the simplest and easiest and it is the only form that can be used with the children's divisions. Therefore, let the teacher develop the simplest and the easiest form of procedure, first, with the simple problems and later, with the more difficult problems and let him introduce the harder form, if at all, only when necessary, the simplest process. The teacher who remarks that short cuts are easier is making no distinction between the method for the learned and procedure for the learner. The question is not a stop long enough to inquire why it is that it is easier, it cannot be used with the young children. If it is so easy, why does no one learn to do the harder, the easier divisions? If it is so easy why should it be learned at all, when the pupil moves ahead to the harder work?

b. *Partition and Division* While preparing for partition division, it is suggested that the children learn the method at the outset. The pupil will have something to do with the division to divide. When he has learned division and the method of discovering the number of smaller equal groups or a number group, he will pass on to the other side and the division and in doing the procedure of dividing into equal groups. In this connection, he will learn that multiplication and division are connected by dividing it by two. Now, if the child has learned to do this, he can engage in partition. Dividing and finding out how many of a number by means of division. The child's own method of division in partition will eventually lead him to the other side of his own generalization regarding the relation between the two ideas.

c. *Remainders in Division* While preparing for partition division, it is suggested that the pupil be given some work

the necessity of having to deal with remainders until he has a chance, in the study of fractions, to learn what to do with one when he gets it. If the suggestion is followed, the pupil will neither be disturbed by the necessity of dealing with remainders while he is learning to divide nor be misled by having to learn a false method of thinking about them. In the first place, the pupil will have enough to do to learn to divide without being troubled by remainders. In the second place, he is inclined to dislocate the remainder and divisor, when in reality the former may be interpreted only in terms of the latter.

To illustrate, when the pupil is required to deal with remainders from the outset, he is inclined to consider the answers in such di-

$$\begin{array}{r} 3 \qquad 3 \qquad 3 \\ 2 \overline{)7} \quad 3 \overline{)10} \quad 4 \overline{)13} \quad \text{In} \\ 6 \qquad 9 \qquad 12 \\ 1R \quad 1R \quad 1R \end{array}$$

each, he secures the answer, 3 and 1 Remainder. In each case, the 1 Remainder is meaningless, because it is kept out of relation with the divisor. Instead of the three divisions producing the same answer, they obviously produce different answers— $3\frac{1}{2}$, $3\frac{1}{3}$, and $3\frac{1}{4}$.

VII. MULTIPLYING BY TENS

The following paragraphs are included to illustrate both the procedure that may be followed in introducing pupils to the idea of multiplying by tens and the emphasis that may be placed upon the idea that tens are multiplied "just like units." Each step of the presentation suggests its own type of practice, which should be provided before the succeeding step is attempted.

The Teaching of Multiplying by Tens

Last year you learned that you can do a number of interesting things with *tens*. Whatever they were, you learned that you can do the same with *tens* that you can do with *units*, so long as you remember that they are *tens* or keep the answers you get in their

proper places. One of the things we must remember is that *ten* goes exactly as you multiply *units*.

The thing to be learned now is how to multiply by *ten*. Do you suppose that is clear? The answer is *yes*. The *ten* goes exactly as you multiply by *units*. The next thing to remember when you are multiplying by *ten* is to keep the answer in *ten's* place. If you just remember to put the answer in that proper place, you can go right ahead and multiply without even thinking of *ones*. Let us see

4193 We multiply by 2 *ones* exactly the way we would multiply
20 by 2. We think, "Two *thousands* are *two*," *Two* *hundreds* are
83860 eighteen," and so on. But look where we write our answer.
When we think, "Two *thousands* are *two*," we write 2 just under the 2, which is in *ten's* place. (We remember that we are multiplying by 2 *ones*.) In order to put our answer in *ten's* place we must write 0 to the right to put it there. Finally we write the 6 right at the beginning, so that we shall not have to think about our answer being *ones*. So we do not multiply any like this

4193 We notice in the beginning that *thousands* are *ones* to multiply by, and that we will carry *thousands* by the 2 *ones*; so we write 0 under 3, and think, "Two *thousands* are six," write 6 under 2, think, "Two *hundreds* are eight *hundreds*," write 8 and remember 1 to carry, and so on.

The Teaching of Multiplying by *Tens* and *Tens*

You have just learned how to multiply by *ones*. Let us now learn to multiply by *units* and *ones*. Suppose we want to multiply 42 by 42.

523 Our multiplier is 42, which is, as you know, 4 *ones* and 2
42 First, we multiply by the 2 *ones*, and start the writing of our answer just under the 2. The product is 84, which is only part of the whole product that we are to find. We have yet to multiply by the 4 *ones*. We will multiply by the 4 *ones* just as though it were a *ones* and start the writing of our answer just under the 4. The product is 20,920. (Notice that this part of the product is not really 20,920 *ones*.) There is no need to write this part of the answer as the rest of the answer keeps this part in its proper place.

We now have two parts of the whole product that we have to

find, 1046 and 20,920. In order to find the whole product we add the two parts, which gives the answer.

First, multiply by units

Second, multiply by tens

Third, add the two part products

Throughout, keep each part of the answer in its proper place.

The Teaching of Multiplying by Hundreds

You already know that you can add, subtract, multiply, and divide *hundreds* exactly the way you add, subtract, multiply, and divide *tens* and *units*. How do you suppose you should multiply by *hundreds*? The answer is easy, isn't it? You multiply by *hundreds* in exactly the same way that you multiply by *tens* and *units*. The only thing to be careful to do is to begin writing the answer you get in the proper place, which is *directly underneath* the multiplier.

734
200
146800

We see at the start that there are no *units* and no *tens* in the multiplier, and that when we multiply, we must multiply by 2 (*hundreds*). Thus we know that our answer must show *hundreds*. Just as when we multiply by *tens*, we now must start writing our answer *directly underneath* the 2, so that our answer will show *hundreds*. Not having any *units* or *tens* in our multiplier, we will get no *units* or *tens* in our answer to hold these places for us, so we start right at the beginning to write a 0 under 0 in *unit's* place, and a 0 under 0 in *ten's* place, and then start our multiplying by the 2 (*hundreds*). "Two fours are eight," "Two threes are six," "Two sevens are fourteen." The two zeros we write at the start place our answer in the places it belongs.

The Teaching of Multiplying by Units, Tens, and Hundreds

You have just learned how to multiply by *hundreds*. Let us see how to multiply by *units*, *tens*, and *hundreds*. Suppose we want to multiply 638 by 425.

638
425
3290
1316
2632
270650

Our multiplier is 425, which is, as you know, 4 *hundred*, 2 *tens*, and 5. First we multiply by the 5 (*units*), and start writing our answer just under the 5. The product is 3290, which is only part of the whole product we have to find. We have yet to multiply by the 2 *tens* and the 4 *hundred*. We now multiply by the 2 *tens* just as though it were 2 *units*, and start writing our answer just under the 2 in order that

this part of the whole product is written, 4315, which really is 4315 *hundreds*.) There is no need to write the zero, as the rest of our work keeps this part of our answer in its proper place.

We now have two parts of our whole product that we have to find, 3452 and 431,500. In order to find the whole product, we add the two parts.

Let us now multiply when we have zero in *units*'s place in the multiplier and need to multiply by *tens* and *hundreds* only.

433	First, we write 0 under 0, so that when we multiply by
540	the 4 <i>tens</i> the product will show <i>tens</i> . We multiply by the
34520	4 <i>tens</i> just as though it were 4 <i>units</i> and start writing
4315	the answer just under the 4. The product is 34,520, which
400020	is only part of the whole product we have to find. We
	have yet to multiply by the 5 <i>hundreds</i> .

We now multiply by the 5 *hundreds* just as though it were 5 *units*, and since that part of our answer must show *hundreds*, we start writing it just under the 5. There is no need to write any zero, as the rest of the work keeps that part of the whole answer in its proper place. The product is 431,500. (Notice that this part of the whole product is written, 4315, which really is 4315 *hundreds*.)

We now have two parts of our whole product that we have to find, 34,520 and 431,500. In order to find the whole product, we add the two parts.

VIII Dividing by Tens

The following paragraphs are included to illustrate both the procedure that may be followed in introducing pupils to the idea of dividing by tens and the emphasis that may be placed upon the idea that tens may be employed in thinking and in computation "just like units." As in the illustrative exercises that have been included under former topics, it is to be understood that each step of the presentation should be followed with sufficient practice to fix it in mind before the work proceeds to the next step.

The Teaching of Dividing by Tens

Last year you learned that you can add, subtract, multiply, and divide tens just the way you add, subtract, multiply, and di-

$$\begin{array}{r}
 4193 \\
 20 \overline{) 83860} \\
 \underline{80} \\
 38 \\
 \underline{20} \\
 186 \\
 \underline{180} \\
 60 \\
 \underline{60} \\
 537 \\
 60 \overline{) 32220} \\
 \underline{300} \\
 22 \\
 \underline{180} \\
 420 \\
 \underline{420}
 \end{array}$$

You think and work like this

"Two in eight, four " Multiply, subtract, bring down.

"Two in three, one " Multiply, subtract, bring down.

"Two in eighteen, nine " Multiply, subtract, bring down.

"Two in six, three " Multiply

Look at the 6 (tens) in the divisor, and since the first figure, 2, in the dividend is less than 6, look at the 32. Think, "Three in thirty-two, five " (Notice where 5 is written) Multiply, subtract, and bring down the next figure. Look at 6 and at 22. Think, "Three in twenty-two, three " Write 3 in the answer, multiply, subtract, and bring down the next figure. Look at the 6 and at the 42. Think, "Seven in forty-two, seven " Write 7

in the answer, and multiply

In dividing by tens, keep your eye on the number in ten's place in the divisor. This is the important number. You know that it is tens, but you can think of it in dividing the same as you think of units in dividing. Be sure to write the answer in its proper place in the quotient.

Teaching What To Do with Zeros in Division

You know that zero is used just to hold a place when there is no number to write in it. In dividing, the zero will often help you to put each part of the answer in its proper place.

$$\begin{array}{r}
 3 \ 0 \\
 70 \overline{) 21420} \\
 \underline{210} \\
 420 \\
 \underline{420}
 \end{array}$$

What is wrong with the answer? The 3 is in its proper place above 4, and the 0 is in its proper place above 0.

This is the right way to work the example

$$\begin{array}{r}
 306 \\
 70 \overline{) 21420} \\
 \underline{210} \\
 420 \\
 \underline{420}
 \end{array}$$

70's in 214? Look at the 7 and at the 21, which tells you, "Seven in twenty-one, three " Write 3 in its proper place, multiply, subtract, and bring down the next number.

$$\begin{array}{r} 7 \\ 74 \overline{) 53578} \\ \underline{518} \\ 17 \end{array}$$

example looks like this

$$\begin{array}{r} 7 \\ 74 \overline{) 53578} \\ \underline{518} \\ 177 \end{array}$$

Since 53 is less than 74, the first quotient will be, 74's in 535. Look at the 7, and at the 5. Think, "Sevens in fifty-three, seven." Write 7 in its proper place, and multiply. (Notice the product.) Subtract. (Notice the remainder.) Bring down the next figure, and the ex-

ample now looks like this

$$\begin{array}{r} 72 \\ 74 \overline{) 53578} \\ \underline{518} \\ 177 \\ \underline{148} \\ 298 \end{array}$$

74's in 177? Look at the 7, and at the 17. Think, "Sevens in seventeen, two." Write 2 in the answer, and multiply. (Notice the product.) Subtract. (Notice the remainder.) Bring down the next figure. The example now looks like this

$$\begin{array}{r} 724 \\ 74 \overline{) 53578} \\ \underline{518} \\ 177 \\ \underline{148} \\ 298 \\ \underline{298} \\ 0 \end{array}$$

Teaching True Quotient Answers

In the examples of division by *tens* and *units* that you have just been working, the quotient answer you found in each case by thinking of the *tens* in the divisor as *units* has been the 'true' quotient answer. Let us illustrate.

$$\begin{array}{r} 6 \\ 76 \overline{) 4788} \\ \underline{456} \\ 22 \end{array}$$

In this example we wish to find how many 76's there are in 4788. To divide, we think of the 7 (*tens*) as 7, and think, "Sevens in forty-seven, six." We write 6 in its proper place and multiply. We notice the product, 456, and see that it is less than 478; so we subtract. We now notice the remainder, 22, and see that it is less than 76. (The product, if it is not the same as the number above it, must be less;

1. 1950年10月1日，中华人民共和国成立，标志着中国历史进入了一个新的纪元。

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The first thing I noticed when I stepped out of the car was the cold. It was a sharp contrast to the warm blanket I had been sitting under. I looked up at the sky, which was a deep, dark blue, and felt a sense of peace. The air was crisp and clean, and I could hear the distant sounds of the city. I took a deep breath and felt a sense of renewal. I had been so stressed and overwhelmed, but now I felt like I was starting over. I looked down at my hands, which were slightly numb from the cold, and felt a sense of hope. I was going to make it through this. I was going to be a doctor. I was going to save lives. I was going to be a hero.

$$\begin{array}{r} 6 \\ 76 \overline{)4940} \\ \underline{456} \\ 380 \end{array}$$

We try 6 by multiplying and noticing that the product is less than the number above it. We are now almost sure that 6 is also the true quotient answer, but not quite sure. We now subtract, and notice that the remainder is less than the divisor. We are now quite sure that 6 is the true quotient answer. We now bring down the next figure.

$$\begin{array}{r} 65 \\ 76 \overline{)4940} \\ \underline{456} \\ 380 \\ \underline{380} \\ 0 \end{array}$$

We again think of the 7 (tens) as 7, and think, "Sevens in thirty-eight, five." We try 5 by multiplying and noticing now that the product is the same as the number above it. This makes us sure that 5 is the true quotient answer. If we wish, we can prove our answer by multiplying 76 by 65, or 65 by 76.

Let us try another example. Suppose we wish to find out how many 48's there are in 2736. We divide.

$$\begin{array}{r} 6 \\ 48 \overline{)2736} \\ \underline{288} \end{array}$$

We think of the 4 (tens) as 4, and think, "Fours in twenty-seven, six." We try 6 as a quotient answer by multiplying noticing that the product, 288, is greater than the number above it. This makes us certain that 6 is not the true answer. We erase and try the next smaller number, 5, as a quotient answer.

$$\begin{array}{r} 5 \\ 48 \overline{)2736} \\ \underline{240} \\ 336 \end{array}$$

We try 5 by multiplying and noticing that the product is less than the number above it. But yet we cannot be entirely sure of 5 until we subtract, and notice that the remainder is less than the divisor. When we do that, we are sure that 5 is the true quotient answer. We bring down the next figure.

$$\begin{array}{r} 58 \\ 48 \overline{)2736} \\ \underline{240} \\ 336 \\ \underline{384} \end{array}$$

We think of the 4 (tens) as 4, and think, "Fours in thirty-three, eight." We try 8 as a quotient answer by multiplying and noticing that the product is greater than the number above it. This makes us certain that 8 is not the true answer. We erase and try the next smaller number, 7, as a quotient answer.

$$\begin{array}{r} 57 \\ 48 \overline{)2736} \\ \underline{240} \\ 336 \\ \underline{336} \\ 0 \end{array}$$

We try 7 by multiplying and noticing that the product is the same as the number above it. This makes us sure that 7 is the true quotient answer. Since there is no remainder, and nothing to bring down, our example is finished. If we wish, we can prove our answer by multiplying 48 by 57, or 57 by 48.

In the exercises you have been taking to find out if the trial quotient answers are the true ones, you have been making your

More About Trial Quotient Answers

Sometimes we have to try more than one trial quotient answer before we can find the true one

$$\begin{array}{r} 4 \quad 6 \quad 5 \quad 4 \\ 27 \overline{)1242} \quad 162 \quad 135 \quad 108 \\ \underline{108} \\ 16 \end{array}$$

We think of the 2 (*tens*) as 2, and think, "Twos in twelve, six." We try 6 by multiplying, and noticing the product. Next, we try 5 in the same way. Next, we try 4 in the same way. We

write 4 in the answer, multiply, notice the product, subtract, notice the remainder, and bring down the next figure.

$$\begin{array}{r} 48 \\ 27 \overline{)1242} \\ \underline{108} \\ 162 \quad 8 \quad 7 \quad 6 \\ \underline{162} \quad 216 \quad 189 \quad 162 \end{array}$$

We think of the 2 (*tens*) as 2, and think, "Twos in sixteen, eight." We try 8 by multiplying, and noticing the product. Next, we try 7 in the same way. Next, we try 6 in the same way.

We write 6 in the answer, multiply, and notice the product.

In dividing by *tens* and *units*, think of the *tens* as *units* and divide. Try the quotient answer by multiplying. If the product is larger than the number above it, try the next smaller number as a quotient answer. Keep on trying the next smaller number until you get the true quotient answer. A quotient answer is the true one if, when you multiply the divisor by it, the product is less than the number above it, or the same as the number above it, and if, when you subtract, the remainder, if there is any, is less than the divisor.

$$\begin{array}{r} 9 \\ 28 \overline{)2436} \quad 252 \end{array}$$

Notice, first, there are no 28's in 24, and the first question is, 28's in 243. Think of the 2 (*tens*) as 2, and think, "Twos in twenty-four; of course, the answer is 12." But since 9 is the largest quotient answer one can use at any one time, do not try 12, then 11, then 10; but use 9 as the first trial answer. In this case, 9 is not the true answer, but 8 is.

$$\begin{array}{r} 9 \\ 17 \overline{)1581} \quad 153 \end{array}$$

Notice, first, there are no 17's in 15, and the first question is, 17's in 158. Think of the 1 (*ten*) as 1, and think, "Ones in fifteen." Since that answer is larger than 9, use 9 as the first trial answer. In this case, 9 proves to be the true answer.

Remember: *Never use a number larger than 9 as the first trial quotient answer.*

Proving That a Remainder Is None

Sometimes in trying quotient answers a person reaches a quotient in multiplying by a trial answer, and so tries to multiply again by the answer, and proves it is a lower trial answer. This is not a good way to tell when he has made such a mistake of his answer is a true answer. Let us see.

$$\begin{array}{r} 7 \quad 9 \quad 8 \quad 7 \\ 34)2788 \quad 306 \quad (202) \quad 228 \\ \underline{238} \end{array}$$

In dividing 2788 by 34, we suppose that 80 is the answer. We try to multiply 34 by 80, and find that the product is 2720. This is less than 2788, so we try a higher quotient. We try 81, and find that the product is 2754. This is less than 2788, so we try a higher quotient. We try 82, and find that the product is 2788. This is the same as the dividend, so we know that 82 is the correct answer. But this trial is not complete until we subtract. Let us see what the example looks like this:

$$\begin{array}{r} 7 \\ 34)2788 \\ \underline{238} \\ 40 \\ 8 \\ 34)2788 \\ \underline{272} \\ 6 \end{array}$$

Look at the remainder. It is smaller than the divisor, so it is less than 34, as it should be, so we know that 82 is the correct quotient answer. We must try a higher quotient than 82. We try 83, and find that the product is 2822, which is more than 2788. Our product, 278, is less than the dividend, 2788. We subtract, and notice that the remainder is less than the divisor. We now know that 82 is the correct quotient answer; so we bring down the next figure, and go on.

Teaching the Use of Zero in Division

You know that zero is used just as much as a figure. In dividing, the zero will often help in putting each part of the answer in its proper place. Let us see.

$$\begin{array}{r} 507 \\ 65)32055 \\ \underline{325} \\ 455 \\ \underline{455} \end{array}$$

The first question is, 65's in 320? I look at the 3 and see the 32, and think, "How many 65's in 32?" I try 5, multiply, notice the product, subtract, and bring down the remainder. Bring down the next figure. You can see that there are no 65's in 45, so we put a 0 in the answer to keep the place. I'm sure you'll know.

the other parts of the answer in their proper places. Bring down the next figure, and complete the division.

65's in 435? Look at the 6 and at the 43, and think, "Sixes in forty-three, seven." Write 7, and multiply

$$\begin{array}{r} 370 \\ 65 \overline{) 4350} \\ \underline{455} \\ 800 \\ \underline{840} \\ 60 \\ \underline{65} \\ 5 \end{array}$$

In this next example the first question is, 65's in 370? Look at the 6 and at the 37, and think, "Sixes in thirty-seven, six." Try 6. Since that is too much, try 5. Multiply, notice the product, subtract, and notice the remainder. Bring down the next figure.

65's in 435? Look at the 6 and at the 43, and think, "Sixes in forty-three, seven." Write 7, and multiply. Notice the product. Since there is no remainder, bring down the next figure, which is zero. This means that there are no units to be divided, and that the answer so far is *seven*, and that there are no units in the answer. Write 0 in the answer to keep the rest of the answer in its proper place.

Teaching Dividing by Hundreds

You have been learning how to divide by *tens*, and you have found out that you divide by *tens* in exactly the same way that you divide by *units*. And you have learned that you need to be careful to put each part of the answer in its proper place, and to try each quotient answer until you find the true one.

Let us see how one divides by *hundreds*. How do you suppose that is done? Perhaps you can guess. You remember, of course, that *hundreds* are added, subtracted, multiplied, and divided just like *tens* and *units*, and that you multiply by *hundreds* just as you multiply by *tens* and *units*. The answer is that one divides by *hundreds* just as he divides by *tens*, and just as he divides by *units*. One thinks of the *hundreds* as *units*, and divides by *hundreds* just as he would if they were *units*. He must be careful, though, to put each part of the answer in its proper place, and to try each quotient answer until he finds the true one.

$$\begin{array}{r} 73 \\ 600 \overline{) 43800} \\ \underline{4200} \\ 1800 \\ \underline{1800} \\ 0 \end{array}$$

We notice at the beginning that there are no 600's in 4, or in 43, or in 438, so our first question is, 600's in 4380. (Notice where we must write our first quotient answer.) We look at the 6 (*hundreds*) and at the 43 (*hundreds*), and think, "Sixes in forty-three, seven." We try 7 by multiplying, noticing the product, sub-

CHAPTER XVI

COMPARISON AND CONTRAST OF THE PROCESSES

Assessment

1. To the extent that pupils understand the number system, they are intelligent about the varied number processes they are called upon to learn.

2. The use of the 'problem' to provide either a routine for the learning of a process or an opportunity to apply the process is limited. The real purpose of the problem is to provide practice in the recognition of general ideas, or, later, illustration of situations of which the general ideas are a part.

3. A general idea once developed serves to unite in thought certain practical situations that otherwise are radically different in character and in setting.

4. The general idea that is portrayed in a problem should be the center of attention, not the problem itself. To this end, the pupil's attention should be directed away from the momentarily interesting and attractive features of the problem situation toward the general idea. The procedure is briefly illustrated by appropriate lesson materials.

5. Lesson materials are used to illustrate the similarities and difference between.

- (a) Addition and multiplication;
- (b) Addition and subtraction;
- (c) Subtraction and division;
- (d) Multiplication and division.

6. Two-step problems are described as means of providing practice in the recognition of general ideas on a level higher than is provided by one-step problems.

7. The additional, or 'higher level,' practice is provided by the necessity to search for the hidden question.

8. The type of instruction that may be given to introduce the two-step problem is illustrated by a variety of exercises.

I. ATTENTION TO ARRANGEMENT

The activities suggested in the preceding paragraphs proceeding may be described as preliminary to arrangement. Through them the pupils are given direction in their studies and arrangements of their studies become gradually more and more systematic.

The studies become more systematic as they proceed in that they depend less and less upon particular suggestions and arrangements and deal more and more with the arrangement of a matter of thought. Instead of merely copying what the teacher's demonstrations through the examples suggest in their studies of arrangement, the pupils are encouraged to think for themselves, the pupils gradually acquire the ability to proceed almost directly, in learning a new principle for a certain relation of arrangement when the relation is presented.

The studies become more systematic as they proceed in that they deal less and less with particular ideas and suggestions and more and more with ideas suggested indirectly as through a central idea -- an idea that suggests a certain way of proceeding of explanation and a standard of what should be done as a common mode of procedure as suggested by the idea. Instead of having to think of each new arrangement as through its appearance when actual demonstration is made, the pupils are enabled in the beginning, the pupils gradually acquire the ability to think of each new arrangement as through its relation to the idea of ten and to the standard of counting with tens.

The progress of pupils may be explained in another way. They are gradually gaining the understanding of the number system, and this system makes them think about large numbers more and more. The following quoted paragraph will help to make this point clear:

Beyond ten, all kinds of objects pass out of the range of direct apprehension. No one can see twenty five objects and

clearly distinguish them from forty-four or forty-six. Only gross differences can be recognized when the number of objects exceeds ten. Ten subjects were shown groups of lines drawn in the middle of a page. Twenty-five short lines scattered irregularly over four square inches could always be recognized as less than thirty-two. If the two groups differed by less than seven, the subjects were irregular in their estimates, sometimes deciding that there were more than twenty-six and sometimes calculating double.

Even twenty-five is a relatively small number. A group of lines which can be distinguished from a group containing 126 lines must contain a number very far removed from 126. Two subjects seemed to be only fairly certain of the contrast between two groups containing 126 and 150 lines, respectively. These facts show clearly that such knowledge as one has of very large numbers is defective and altogether dependent on the use of some kind of a tally system. If a well-arranged tally system, such as the Arabic numeral system, can be used, thinking about large numbers can be made precise in spite of the fact that the group which a particular tally represents in itself has large to be appended directly. Within the tally system, 126 and 127 are clearly distinguishable, while the groups to which the two numbers refer are quite indistinguishable. If we wish to deal with large numbers, it is evident that we can be precise only when we use a tally system.¹

Of necessity, the pupils must rely less and less upon experience with the concrete and depend more and more upon the system of dealing with tens "just like units." Thus, in

53

the easy subtraction, $\overline{53} - 28$, the pupils do not attempt to set before themselves 53 objects, count off 28 and take them away, and count the number remaining. Such a task would be tedious and greatly subject to error. Instead, the pupils take units from units, and, when that task is completed, they take tens from tens in exactly the same manner. In

¹ C. H. Judd, *Psychological Analysis of the Fundamentals of Arithmetic* (Department of Education, The University of Chicago, Chicago, 1927), p. 72.

the division. By the use of the method of grouping them into three, and subdividing them into a long and a shorter one (instead of the groups of three, then two, then one, and so on), the units at the instant of each problem. These "units" of problems are rendered easy and clear for the system enables one to remember larger quantities in a more stable group and to deal with the problem in a more easy way.

II. THE LINE OF PROBLEMS

'Problem' — a situation that describes and leads to a specific concrete situation. The number combinations and problems that the subject has to learn. When no used, the problems are given and the combinations are given, as a means of presenting to the minds of the pupils the learning of the combinations and of supplying an interest in, and a feeling of control of the combinations before they are learned. Problems are at times used after a particular combination has been learned as a means of presenting a particular application. The theory underlying such a use of problems appears to be that the pupil first learns a general solution, let us say, as something to do with the formula of addition may be applied. In this case the problem is used to provide a specific for the formula. In other, it is used to provide an opportunity for applying what has been learned; in both, the problem is a means of presentation for the learning of a process that will be useful in the future (so it appears to be considered) a useful performance or useless performance. In both cases the problem is a situation it describes, in the sense of situation and the process that is to be learned or that has been learned is a matter of secondary importance. Problems are of value in its own right.

The foregoing paragraph is not intended to be a definition of the

uses named. To the extent that problems can be so used, such uses are without doubt justifiable ones. The paragraph is intended, rather, to indicate the limitations of such uses, and to call attention to the narrowness of any procedure that seeks to hide from the consideration of pupils the meaning and the value of the number system.

While the problems that are appropriate as exercises following the various processes illustrated in the foregoing chapter, for example, will have as one of their purposes the presentation of concrete illustration of the uses that can be made of such processes, they should be intended also as a means of furthering the purpose of problems stated in Chapter XI. The purpose of problems in the early stages of arithmetic has been described as that of giving opportunity for practice in recognizing such general ideas as addition, etc., in familiar situations, in order that these ideas may be clarified and enlarged. Since the pupils who are engaged in the task of learning the processes illustrated in the foregoing chapter are still in the early stages of arithmetic, we may say that the purpose of the problems that accompany such processes is to further the development of the general ideas of addition, subtraction, multiplication, and division.

III. THE IMPORTANCE OF GENERAL IDEAS

The importance of general ideas has already been discussed. What may be said here will, therefore, be in the nature of further illustration of their importance.

To many pupils, addition means merely something to do. They have a word in their vocabulary — 'addition' — that has a very narrow meaning. When told to add, either by their teacher, or by some sign or clue in an exercise, such as $+$, "how many altogether," "sum," etc., they can respond by adding. 'Addition' then, means for them to perform in a given way. Many pupils can so perform without having any idea of the meaning of the performance or what really has been accomplished by such performance. The writer

has encountered pupils in the fifth grade, who, when presented with "problems" involving the carrying of numbers only, have raised their heads in inquiry. "What does carry mean 'add'?" When such pupils are told to add they may proceed to add, or, if told to subtract or multiply they may proceed with equal assurance to perform these operations. Such pupils have missed the rich background of experiences that make the term 'addition' meaningful. They may use addition as a means of getting an answer (which answer is meaningless) only after numerous times under the direction of the teacher. They cannot use addition as an action that builds up relation a series of concrete experiences that, in all other respects, are separate and unrelated items of experience.

Our discussions of 'problem-solving' have suggested that the pupil needs practice in reorganizing the general ideas of the processes in a variety of fanciful situations. We need to bear in mind that such a word situation, though familiar, may, because of their variety, be rather unhelpful rather than helpful unless the pupil is already in possession of the general ideas reorganized and is consciously seeking an expression of them in the various situations. The general idea, which may be illustrated by many situations, serves as a means of bringing the many situations together in a common bond of relationship. To illustrate:

Before one groups together cases in which money is earned and cases in which money is otherwise acquired, one should clearly recognize the fact that a money situation when it involves accumulation is implied as existing in the mind of the reader of the problems. The word 'earned' and the phrase 'received as a gift' are not identical and do not involve suggestion except in the thinking of an individual who has experienced the two distinct concrete ideas into abstract mathematical ideas. When pupils begin to study money relations, they do not have the general idea of addition, and they cannot bring

* See G. T. Burwell and Lester John. The Psychology of Mathematics. Chap. VI. (Department of Education, The University of Chicago, 1931.)

regarded as those involving 'earning' and 'receiving as a gift' except as they illustrate the more general notion of accumulation.¹

In the cases mentioned in the foregoing quotation, if the pupil is resorting to the problems that involve 'earning' and 'receiving as a gift' for opportunity to 'apply' the process of addition, which to him is but a process to perform, his interest will center in the two quite different situations and their very differences may obscure the opportunity to apply addition. If, on the other hand, he resorts to them with the various general ideas in mind for the purpose of discovering one or the other of the ideas illustrated by the situations, his interest will center in the particular general idea that appears in the two situations in question and that brings them together in thought.

In a study of the "effect of unfamiliar settings on problem-solving," Brownell and Stretch have discovered some data that "suggest that problems may be so difficult for children that the addition of an unfamiliar situation does not materially alter children's procedures in dealing with them . . . Likewise, these data suggest that problems may be so easy for children that the presence of an unfamiliar setting fails to obscure from them the arithmetical relationships involved."² Perhaps this is only another way of stating that when the general idea -- in this case, the idea of 'on the average' -- is somewhat obscure, it is about equally obscure in the situations involving the number of words spelled in a test and "lots of graks"; and that when the general idea -- in the latter instance, the ideas of addition and subtraction -- is fairly familiar, it may be recognized with equal ease in a situation dealing with a "Russian serf," "pushnas," and "chukets," and in one dealing with "school children," "Health Week," and "making posters."

¹ Judd. *Op. cit.*, p. 94.

² W. A. Brownell and L. B. Stretch. *The Effect of Unfamiliar Settings on Problem-Solving* (Duke University Press, Durham, 1931), p. 45.

The fact that it may be the tendency of the general ideas of mathematics that already are in the mind of the stage in problem and problem-solving may be emphasized in the following paragraphs:

The question which is here presented is, "What are the comparative merits of mathematics and of mathematics as a means for developing four number ideas and children's perceptions of operation?" Is there any danger that the majority of the children only problem-solving which are really simple, and experience, the tracing of number concepts from problem-solving imagery may be unduly delayed? Is the child's mental imagery may it not be possible, from the point of view of mathematics outcome, to begin solving early the use of number and mathematics in problems? Is it not possible that the use of such mathematics settings might have the effect of emphasizing the notion that numbers are essentially in themselves and that their relations are determined by the nature of the numbers themselves rather than by the character of the objects which they represent? Additionally we answer as to the second question, "What are the relations in the data of the present mathematics that is an answer of an answer does not make the question as a whole a problem. The role played by mathematics as organized for mathematics problem-solving will not be unduly delayed, and the ultimate outcome of mathematics should be, since it is based as much consideration as to how mathematics can be used in the lives of children's interest and of the difficulty of problem-solving."

IV. EMPHASIZING THE CONCRETE IDEAS

The activities already suggested by the illustrations in this book are such as to call the attention of pupils from the beginning of their work. It has been understood that the general ideas of mathematics, for example, appear as a part of many circumstances of their lives, and that, if the pupil is to be prepared to use the idea so that he can learn that it is a part of many circumstances, he will have the mathematics as a part of

* Brownell and Smith, *op. cit.* p. 100.

abstraction that he does not possess when he first comes to school. It has been recognized that if the pupil is to be expected to give his attention to a given idea of procedure, or arrangement, or grouping, such as is designated, for instance, by the name 'addition,' he will need to have someone direct his attention away from other interesting features of situations and to help him to notice the procedure, or arrangement, or grouping, whenever things are 'put together' or 'thought together.' The process of abstraction is one of analysis, of neglect of certain features of the situation, and especially of concentration upon other features. From the outset, the teacher has made analysis for the pupil and has assisted him in making analysis by emphasizing and holding up for special study and discussion the arrangements of 'putting together,' 'taking things away,' 'equal groups,' etc. In the beginning, the idea of taking away, for example, was demonstrated, discussed, described, and, later, given the name 'subtraction.' Later, the pupils engaged in taking away and thinking away, following the teacher's directions and questions. Still later, the pupils were presented with a number of situations (problems) that served to illustrate the idea of taking away, and, in dealing with these situations, the pupils were instructed and guided in looking for and discovering the idea of taking away. Moreover, by frequent reviews of what had been learned, the idea of taking away was given renewed emphasis. The following exercises illustrate the means of emphasizing the idea of taking away in contrast with the idea of putting together in the reviews. By means of such contrasts the pupil learns how to look for distinctions and how to make distinctions.

Illustrative Exercises Teaching Pupils to Know Things Exactly

1. Susan had 4 books at school and 5 books at home. How many books did Susan have?

Of course, we could say that Susan had 4 books and 5 books. But we have learned that it is better to say that she had 9 books, because then we know *exactly* how many books she had.

2. Henry had 5 marbles as he picked them up scattered on the ground. On the way 3 marbles slipped out. How many marbles did he have left?

Of course, we could say that Henry then had 2 marbles, but that does not sound just right. We have learned that it is better to say that Henry then had 8 marbles, because then we know exactly how many marbles he had left.

In the first problem, we have learned to add like this: $5 + 3 = 8$. Adding is a way of finding out exactly what we want to know.

In the second problem, we have learned to subtract like this: $8 - 3 = 5$. Subtraction is another way of finding out exactly what we want to know.

Addition is a way of finding out exactly, and subtraction is a way of finding out exactly. Addition and subtraction are exactly alike. That is not quite true, is it? It has no sense to think addition and subtraction are the same in one way, and in another way they are different. The next two topics will tell you how they are different.

Teaching Putting Things Together

Nancy has 6 crayons in her pencil box and 3 crayons on her desk. How many crayons does she have?

How do we find out exactly how many she has? We can know that we find out by adding, like this: $6 + 3 = 9$.

putting 6 crayons and 3 crayons together. Actually we can put them together by piling the 6 crayons and the 3 crayons together in one pile. We just make believe that we can, we just pretend we put them together. In order to find out how many crayons Nancy had, we put them together by adding, as well as the same as thinking them together.

What is addition? Addition is a way of finding out exactly by putting things together or thinking things together. Addition answers the question, 'How many altogether?'

Thinking Taking Things Away

George has 7 cents. How many cents would he have left if he should spend 3 cents for candy?

How do we find out exactly how many cents George would have left? Of course, you know that we find out by subtracting, like

7

minus 3. We find out by taking 3 cents from 7 cents. Actually

4

we would not have to make George spend 3 cents in order to find out, would we? We just imagine that 3 cents are taken away; we just pretend. We just *think* 3 cents from 7 cents. That is what is meant by 'subtraction'. Subtraction is a way of finding out exactly by taking things away or by thinking things away.

Taking away is subtraction. Putting together is addition. That is the way they are different. Do you remember how they are alike?

V. SIMILARITIES AND DIFFERENCES

Enough has been said in our previous discussions, and the available literature* is sufficiently complete and explicit, to remind the reader of the essential relations and contrasts between the general ideas of addition, subtraction, multiplication, and division. The reader's interest is not in how he may remind himself of such relations and contrasts, but in how he may call them to the attention of pupils. The following paragraphs are included in order to suggest a means by which the pupil may be repeatedly reminded of relations and contrasts. The point emphasized is that the pupil will need not merely to review a thing when he has learned it, but also to review it in its relations and contrasts with other things he has learned.

Comparing Addition, Subtraction, Multiplication, Division

The following pages, which compare and tell the difference between Addition and Multiplication, Addition and Subtraction,

* See C. T. Russell and C. H. Judd, *Summary of Educational Investigations Relating to Arithmetic*. Chap. V. (Department of Education, The University of Chicago, Chicago, 1925); also Judd *Op. cit.*, Chap. V.

phases. You have to look carefully at the first part of the problem to learn what particular kind of problem it is.

The first part of Problem 1 tells *separately* how many pieces of candy were made the first week, how many the second week, how many the third, and how many the fourth. Because the number for each of the four weeks is told *separately*, you know that Problem 1 is an *addition* problem. Whenever *how many* or *how much* for more than one case or more than one thing is told *separately* for each, and the problem is a *put-together* or *think-together* problem, it is an *addition* problem.

The first part of Problem 2 does not tell *separately* for each week how many pieces of candy were made, but gives the number that is the same for all. Notice it says "28 pieces each week for 4 weeks." Because the number for each of the four weeks is not told *separately*, but is the same for all, you know that Problem 2 is a *multiplication* problem. Whenever *how many* or *how much* for more than one case or more than one thing is *not* told *separately*, but is the same for all, and the number of cases or things is told, the problem is a *multiplication* problem.

Notice how Problem 3 tells *how much separately*, and how the last part tells you that it is a *think-together* problem. By "how much *separately*" you can tell that it is an *addition* problem.

3. For lunch Susan bought a salad for 10 cents, a bowl of soup for 5 cents, some vegetables for 5 cents, bread and butter for 4 cents, and milk for 5 cents. How much did her lunch cost?

Notice how Problem 4 tells *how much not separately*, but *the same for all*, and gives the number of cases or things. By "how much *the same for all*" you can tell that it is a *multiplication* problem.

4. For lunch Susan spent 32 cents each day for 5 days. How much did her lunches cost?

How Addition and Subtraction Differ

You have learned that addition and subtraction do not mean the same. Addition means *put things together* or *think things together*, and subtraction means *take things away* or *think things away*. Addition is used to find the total, or sum, or how much or how many altogether, and subtraction is used to find the differ-

ance, or how much or how many left, and also to find out which more or less, greater or smaller, etc.

1. There are 35 children in the fourth grade of the Lincoln School, and there are 45 children in the fifth grade. How many children are in both of these grades?

2. There are 35 children in the fourth grade of the Lincoln School, and there are 45 children in the fifth grade. How many more children are there in the fifth grade?

Problems 1 and 2 both tell how many separately. But when you know they are not multiplication problems. Notice the question at the end of the problems, and how they both give answers to the kind of problem each one is.

3. Henry had \$5.75. He made \$2.50 by mowing lawns. How much did he then have?

4. Henry had \$10.25. He spent \$4.50 for a pair of shoes. How much did he have left?

Problem 3 tells how much separately and asks how much altogether. Problem 4 tells how much altogether and asks how much was taken away, and asks how much was left.

When a problem tells you to find change together and is not multiplication, it is an addition problem. When a problem tells you to think things away to find out how much is left, or when it tells you to find out about more or less, it is a subtraction problem.

How Subtraction and Division Differ

Subtraction is sometimes something like division, but they are never exactly the same. They are somewhat alike when they both deal with wholes and parts. In subtraction sometimes the whole is given, and one part is given, and you have to find the other part by subtracting. In division sometimes the whole is given, and the number of parts is given, and you have to find how many in each part by dividing, or the whole is given and how much or how many in each part is given, and you have to find the number of parts by dividing. (Notice the word *each* in the sentence telling about division.)

1. There are 637 children in the Central School. There are 351 children in the lower grades. How many children are there in the upper grades? (Notice that the whole is given (637) and one part is given (351), and that you are to find the other part.)

2. There are 18 teachers in the Central School. Each grade has the same number of teachers. There are 6 grades in the school. How many teachers are there in each grade? (Notice that the whole is given (18), and the number of parts is given (6), and that you are to find how many in each part.)

3. There are 18 teachers in the Central School. There are 3 teachers in each of the grades. How many grades are there? (Notice that the whole is given (18), and the number in each part is given (3), and that you are to find the number of parts.)

How Multiplication and Division Differ

Multiplication and division are alike in some ways, but, as you remember, each is the opposite of the other. The word 'each' usually tells that a problem is either a multiplication or a division one, and it is easy to tell one from the other.

1. There are 18 classrooms in the Central School. In each classroom there are 6 sections of blackboard. How many sections of blackboard are there in all of the classrooms?

2. Helen bought 12 colored pencils for her drawing kit. Each pencil cost 6 cents. How much did all the pencils cost?

You can tell that Problems 1 and 2 are multiplication problems. Problem 1 tells the number of parts and how many in each part. Problem 2 tells the number of parts and how much in each part. In Problem 1 you are to find how many altogether, and in Problem 2 you are to find how much altogether.

3. There are 108 sections of blackboard in the 18 classrooms of the Central School. In each classroom there are the same number of sections. How many sections of blackboard are there in each classroom?

4. There are 108 sections of blackboard in the Central School. There are 6 sections in each classroom. How many classrooms are there?

5. Helen bought 12 colored pencils for her drawing kit for 72 cents. How much did each pencil cost?

6. Helen bought some colored pencils for her drawing kit for 72 cents. Each pencil cost 6 cents. How many pencils did she buy?

You can tell that Problems 3, 4, 5, and 6 are division problems. Problems 3 and 5 tell the whole amount, or how many or how much

altogether (104 and 72 people) and after 1 or 2 repetitions of games 1 & 2 (see 12). You are to find how many of these people are of each group (say 3, 4 and 5 and the whole amount of the 104 and 72 people altogether (104 and 72 people) and after that you are to find how much (6 people) in each group. If the total is 104 and 72 people altogether the number of people

[illegible][illegible][illegible]

At first glance the two-step problem, seen in the symmetrization of being nothing more than two consecutive problems in one and of providing the pupil the same kind of procedure for both has been having with problems of the same type. In the one-step problem, he must recognize one value as the unknown. In the two-step problem, he must recognize two values. Hence it appears that the two-step problem is different only in the increased level of gives a double amount of practice. The only reason for the increase that appears on the surface in the difficulty of determining values and performing two operations in a series rather than one of one. Accordingly, the two-step problem is being introduced as a means of providing the pupil with practice

practice and practice of a more difficult kind on the same level as the practice he has become accustomed to receiving. An additional justification for the two-step problem is the fact that in many a situation in actual life more than a single operation is necessary.

The two-step problem does provide practice in double amounts, and it does provide an introduction to many concrete situations that have a practical importance, but to explain it merely as a double problem is to omit from its description the statement of a very important element. The one-step problem presents in a single situation a single idea; the two-step problem presents in a single situation -- not two situations -- two ideas. To state the matter in another way, in the one-step problem, a single question is involved and a single question is asked; in the two-step problem, two questions are involved and a single question is asked, leaving the other question to be discovered by the pupil. Herein is the fundamental difference between the two types of problems that in some manner or other must be called to the attention of pupils.

VII. PRACTICE ON A HIGHER LEVEL.

Because one of the questions that are involved in the two-step problem is not explicitly stated, the pupil is required to examine the situation presented by the problem in order to discover the question for himself. He must, as it were, read between the lines in order to discover the question. To do this, he will have to come to the task of attacking the two-step problem with certain general ideas already fairly well in mind. Since the problem does not tell him what ideas are present and leaves one of them unsuggested by a question, the pupil must attack it with the conscious purpose of discovering what is hidden. In other words, the general ideas that are to be illustrated by the two-step problems not only must be well enough known to be recognized when they have an obvious description, as in the one-

step problem, but also served to us. It is not to be forgotten available both as a means of interpreting the situation and as a means of suggesting the solution procedure. This case demands that the two-step problem suggests some question to a new level, and makes of the problem a new source of training. To be sure, it is more difficult to give a solution, for example, when it appears as a subtraction with no other, or a like operation, than it is in the preceding situation when it appears alone. But this is not the whole story. The two-step problem and only previous individualized practice in recognizing the general ideas of addition, subtraction, multiplication, and division and their interrelationships, but also requires the recognition and use of these ideas as means of attack and of interpretation. The two-step problem becomes an aid in interpretation. The two-step problem is to provide an added emphasis to the efforts the teachers have been making to suggest upon the ground the meaning of the general ideas.

The following exercises are presented to illustrate the type of instruction the teacher may practice with regard to two-step problems, the kind of training and practice such problems may give, and the essential points of attention on such problems upon which the pupils must be led to sustain their attention.

Illustrative Teaching of 'Two-Step' Problems

You have learned to solve the problems that are known as one-step problems; that is, problems in which you have to do *one thing* — either add, subtract, multiply, divide, or use a formula. The problems you are now to learn to solve are known as *two-step* problems. Two-step problems are the ones that ask you to do *two things*. You could call them *double* problems, or *two-step* problems, because each is really two problems in one. Let us see. Here is a two-step problem.

1. Billy sold 35 papers and James sold 61. How much did they both get for them if they sold the papers at 4¢ each?

Thus is the way the problem breaks, if we make two problems out of it. (Read 1a and 1b together.)

1a. Holly sold 55 papers and James sold 61. How many papers did they both sell?

1b. Holly and James sold 116 papers at 4¢ each. How much did they get for them?

Of course you know how to solve Problems 1a and 1b. Let us solve them and compare the way they are solved with the way Problem 1 is solved.

(1a)	55	(1b)	116	(1)	25
	61		34		61
	116 papers		\$4 04		116
					04
					\$4 04

Problem 1 is an Addition-Multiplication problem, or an AM problem.

2. John spent 60 cents for marbles. He paid 2 cents for each marble. He divided them among 6 boys. How many marbles did each boy get?

Let us make Problem 2 into two problems.

2a. John spent 60 cents for marbles. He paid 2 cents for each marble. How many marbles did he buy?

2b. John bought 30 marbles. He divided them among 6 boys. How many did each boy get?

Compare the way Problems 2a and 2b are solved with the way Problem 2 is solved.

	30 marbles		5 marbles		30		5 marbles
(2a)	2)60	(2b)	6)30	(2)	2)60		6)30
	6		30		6		30
	0				0		

Problem 2 is a Division-Division problem, or a DD problem.

3. Joe made \$45.80, and spent \$10.80. How much does he need in order to make a payment of \$75.00 on a Ford?

Let us make Problem 3 into two problems:

3a. Joe made \$45.80, and spent \$10.80. How much does he have left?

3b. Joe has \$35.00. How much does he need in order to make a payment of \$75.00 on a Ford?

Compare the way Problems 3 and 4 are solved. Problem 3 is solved

(3a) 845 00	(3b) 824 40	(3c) 845 00	(3d) 845 00
10 00	10 00	10 00	10 00
835 00	814 40	835 00	835 00

Problem 3 is a Subtraction-Extraction problem. It is solved by subtraction.

4. A farmer had 931 bunches of corn. He sold 446 bunches at the root into bags. 2 bunches of corn are in each bag. How many bags did he have?

Let us make Problem 4 into two problems.

4a. A farmer had 931 bunches of corn. He sold 446 bunches. How many bunches did he have left?

4b. A farmer put 446 bunches of corn into bags. 2 bunches in each bag. How many bags of corn did he have?

Compare the way Problems 3 and 4 are solved. Problem 4 is solved

(4a) 931	(4b) 2'446	(4c) 931	(4d) 931
446	4	446	4
446 bunches	4	446	4
	4		4
	4		4
	4		4
	4		4

Problem 4 is a Subtraction-Extraction problem. It is solved by subtraction.

5. Mr. Brown sold celery at 5 cents a bunch. He had 87 bunches. If he trimmed it nicely he could get 12 cents a bunch. How much extra did he make on 87 bunches nicely trimmed? How much extra did he make by trimming it?

Let us make Problem 5 into two problems.

5a. Mr. Brown sold celery at 5 cents a bunch. He had 87 bunches. If he trimmed it nicely he could get 12 cents a bunch. How much extra did he make on a bunch of celery by trimming it?

5b. Mr. Brown made 5 cents extra on 87 bunches by trimming it. How much extra did he make?

Compare the way Problems 3a and 3b are solved with the way Problem 3 is solved.

(3a) 12	(3b) 87	(3) 12	87
9	93	9	93
<u>21</u>	<u>\$2.61</u>	<u>21</u>	<u>\$2.61</u>

Problem 3 is a Subtraction-Multiplication problem, or an SM problem.

6. Henry is working to make enough money to buy a tennis racket that costs \$3.50 and a ball that costs 75¢. He is paid 25¢ an hour for each hour he works. How many hours will Henry have to work to make enough money to pay for them?

Let us make Problem 6 into two problems:

6a. Henry is working to make enough money to buy a tennis racket that costs \$3.50 and a ball that costs 75¢. How much will Henry have to pay for both of them?

6b. Henry is working to make enough money to buy a tennis racket and ball. They both cost \$3.25. Henry is paid 25¢ for each hour he works. How many hours will he have to work to make enough money to pay for them?

Compare the way Problems 6a and 6b are solved with the way Problem 6 is solved.

	25 hours		25 hours
(6a) \$3.50	(6b) .25\$3.25	(6) \$3.50	.25\$3.25
75	50	75	50
<u>\$3.25</u>	<u>1.25</u>	<u>\$3.25</u>	<u>1.25</u>
	<u>1.25</u>		<u>1.25</u>

Problem 6 is an Addition-Division problem, or an AD problem.

7. Mary wants to buy a hat that costs \$3.50 and a dress that costs \$5.25. She now has \$6.20. How much more money does she need?

Let us make Problem 7 into two problems:

7a. Mary wants to buy a hat that costs \$3.50 and a dress that costs \$5.25. How much do they both cost?

7b. Mary has \$6.20. She wants to buy a hat and a dress that cost \$8.75. How much more money does she need?

Compare the way Problems 7a and 7b are solved with the way Problem 7 is solved:

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(7a) \$3.50

5 25

\$3.75

(7b) \$5.75

6 25

\$2.50

(7c) \$3.50

5 25

\$3.75

(7d) \$5.75

6 25

\$2.50

Problem 7 is an Addition-Subtraction problem. It is a 2-2 problem.

8. Sam works 5 hours each day. His mother pays him \$3.50 for each day he works. Last month Sam worked 26 days. How much did he make last month?

Let us make Problem 8 into two problems:

8a. Sam works 5 hours each day. His mother pays him \$3.50 for each day he works. How much does he make each day?

8b. Sam makes \$3.50 each day. Last month he worked 26 days. How much did he make last month?

Compare the way Problems 8a and 8b are solved with the way Problem 8 is solved:

(8a) \$.40

8
\$3.20

(8b) \$3.20

20
18 20
64 0
\$62.20

(8c) \$.40

8
\$3.20

(8d) \$3.20

20
18 20
64 0
\$62.20

Problem 8 is a Multiplication-Multiplication problem. It is a 2-2 problem.

9. Jane gave \$20 to help buy a radio. Harry gave \$15 and John gave as much as Jane and Harry. How much did they all give?

Let us make Problem 9 into two problems:

9a. Jane gave \$20 to help buy a radio and Harry gave \$15. How much did they both give?

9b. Jane and Harry gave \$35 to help buy a radio. John gave the same as Jane and Harry both gave. How much did they all give?

Compare the way Problems 9a and 9b are solved with the way Problem 9 is solved:

(9a) \$20

35
\$55

(9b) \$55

55
\$110

(9c) \$20

35
\$55

(9d) \$55

55
\$110

Problem 9 is an Addition-Addition problem. It is a 2-2 problem.

10 Frank sold 45 dozen ears of corn at 20¢ a dozen. With the money he bought young chickens that cost 50¢ each. How many chicks did he buy?

Let us make Problem 10 into two problems.

10a Frank sold 45 dozen ears of corn at 20¢ a dozen. How much did he get for the corn he sold?

10b Frank spent \$13.50 for young chickens. He paid 50¢ for each chick. How many chicks did he buy?

Compare the way Problems 10a and 10b are solved with the way Problem 10 is solved.

			27				27
(10a)	45	(10b)	50¢	\$13.50	(10)	45	50¢
	20		100			20	100
	\$13.50		3 50			\$13.50	3 50
			3 50				3 50

Problem 10 is a Multiplication-Division problem, or an MD problem.

Teaching How to Work Two-Step Problems

You have noticed that each of these ten two-step problems is in reality a double problem, that each is really two problems in one. Of course, in each only one question is asked, and the one question makes it seem as if there is only one thing to do. But have you noticed in working each problem that you yourself had to ask a question that the problem did not state and answer that one, and then pay attention to the question that the problem did state and answer that one? So, in reality, each two-step problem has two questions to be answered, but one of these questions is not stated in the problem and you yourself have to ask it.

In order to work a two-step problem, you yourself must ask one of the questions.

Each two-step problem is two problems in one. The two problems are not stated as two problems, but stated together as one. You yourself must make it into two problems. In other words, you must read between the lines some words that are not stated, making the problem read like two problems, and then work each one of them. Each part of a two-step problem is very easy to work. The only thing that may be hard to do is to make the problem read like two problems.

In order to work a two-step problem you yourself must make at least two problems. You yourself must ask the hidden question.

You must read between the lines.

Turn back to the two problems that have been assigned. Read and notice how each one can first be solved after two problems by reading between the lines. Let us take notice of the two problems, and see how to read between the lines.

1. Billy sold 25 papers and James sold 61. How many did they both get for them if they sold the papers at 4 cents each?

We must read the whole problem carefully, so that we will know exactly what it is about. When we understand, we shall be working between the lines, making the problem into two problems, and solving them as we go along. Notice.

Billy sold 25 papers and James sold 61. (How many did they both sell?) How much did they get for them if they sold the 86 papers at 4 cents each?

$$\begin{array}{r} 25 \\ 61 \\ \hline 86 \end{array} \qquad \begin{array}{r} 100 \\ 328 \\ \hline 328 \end{array}$$

2. John spent 60 cents for marbles and paid 2 cents for each marble. He divided them among 6 boys. How many did each boy get?

We must read the whole problem carefully, so that we will know exactly what it is about. When we understand, we shall be working between the lines, making the problem into two problems, and working them as we go along. Notice.

John spent 60 cents for marbles and paid 2 cents for each marble. (How many marbles did he buy?) He divided them (the 30 marbles) among 6 boys. How many did each boy get?

$$\begin{array}{r} 30 \\ 2)60 \\ \hline 6 \\ 0 \end{array} \qquad \begin{array}{r} 5 \\ 6)30 \\ \hline 30 \end{array}$$

Read Problems 3 to 10. Read between the lines, making each part of each problem as you go along. In each problem, as you read it, look for the hidden question. You yourself must ask the hidden question.

The Ten Kinds of Two-Step Problems

There are only four things to do in order to work any of the ten two-step problems that have been given — add, subtract, multiply, and divide. In each problem you are asked to do two of them. There are ten different combinations of the two things:

AA	Addition-Addition	See Problem 9.
AS	" " " "	See Problem 7.
AM	" " " "	See Problem 1.
AD	" " " "	See Problem 6.
SS	Subtraction-Subtraction	See Problem 3.
SM	Subtraction-Multiplication	See Problem 5.
SD	" " " "	See Problem 4.
MM	" " " "	See Problem 8.
MD	" " " "	See Problem 10.
DD	" " " "	See Problem 2.

Getting Practice with These Problems

There are three ways of getting good practice with these problems.

1. Read each problem carefully, so that you will understand what it is about. Then read it again, *reading between the lines*, making it into two problems.

2. Read each problem carefully, so that you will understand what it is about. Then tell what kind of a problem it is — an AA, AS, AM, AD, SS, SM, SD, MM, MD, or a DD problem.

3. Read each problem carefully, so that you will understand what it is about. Then read it again, *reading between the lines*, making it into two problems, and working the two as you read along.

Practice the way your teacher tells you. Practice upon most of them the third way named.

CHAPTER XIII

THE STUDY OF PARTS

SUMMARY

1. The idea of the fraction is the idea of the part. It is developed and clarified through a systematic study of parts.
2. Incidental contact with parts leads to familiarity with the name of fractions, but not to an acquaintance with the characteristics of fractions.
3. The fraction is an idea of relative fractions is a misconception.
4. Such type idea is an important element in the full meaning of the fraction.
5. Fractions must also be dealt with in relation to each other. The full meaning is necessary for an intelligent handling of such relations.
6. The relational idea of undivided fractions seems to be very familiar if the relations between fractions are to be explained without distraction and confusion.
7. The equality of size in fractional divisions is a fundamental idea. Attention both to size of parts and a number of parts is necessary to a complete comprehension of any given fractional part.
8. Lesson materials are used to illustrate the study of a fraction:
 - (a) As one or more of the equal parts of a unity or group.
 - (b) As composed of, and as related to, equal parts into which it may be divided.
 - (c) As related to other parts in respect to size and number.
9. Decimals embody the familiar ideas of size and a number of parts represented by the familiar decimal system of notation. The fundamental operations are presented as extensions of such operations with whole numbers.
10. Percentage is a special study and use of the relationship. Lesson materials are used to illustrate how the study of percentage may be introduced.

The idea of the fraction is the idea of the part. It is developed and clarified through a systematic study of parts. Just as groups have to be studied in a systematic and orderly way as a means of developing and clarifying the ideas of number, so must parts be studied in a systematic and orderly way as a means of developing and clarifying the ideas of fractions. The methods used in the study of groups suggest the methods to be followed in the study of parts. Parts may be studied (1) by giving attention to first one typical and commonly used part, then another, separately; (2) by separating one part after another into the smaller constituent parts of which each is composed, and (3) by the comparison of parts. Just as the study of groups did not consist of the study of the manipulation of the numerals that stand for them, so the study of parts does not consist of the study of the manipulation of the fractional expressions. As the numerals are used as a language of expressing number ideas and of keeping the thinking about groups systematic, so must the fractional expressions be used as a language of expression and of thinking. It must be kept in mind that it is parts that are to be studied, not primarily the way the fractional expressions may be written and used.

I EARLY INTRODUCTION TO FRACTIONS

From the time the child has entered school he has been learning to use certain fractional expressions in a manner that for all practical purposes has been sufficiently precise. He has learned to speak of half a day, half a pint, half a yard, half a stick of candy, etc., and, insofar as his needs and experiences are concerned, he has come into possession of the ideas for which each expression stands. He has been absent from school, for example, half a day; and he has been sent to the store to buy and bring home half a pint of milk. The girl has bought at the ten-cent store half a yard of ribbon for her doll; and has broken her stick of candy to give her friend half. In school, half of the class has been

instructed to go to the board, or report her conclusions, and half to remain at their desks, and so on.

Coincident with the pupil's satisfaction in discovery, he has been introduced to the fraction as a way of solving a division. A number must be divided into six equal parts and he learns to speak of finding one-sixth of the number as dividing by six. The directions "find $\frac{1}{2}$ of 24, $\frac{1}{3}$ of 36, $\frac{1}{4}$ of 72, etc. — have come to mean, divide 24 by 4, 36 by 3, 72 by 6, etc. The pupil is not unacquainted with the idea of the fraction when he begins his systematic study of parts. Having become accustomed to using words of the fractions both in speaking and writing, he is able to approach the systematic study of parts with some degree of familiarity with what is to be called to his attention.

II. THE FRACTION AS AN IDEA OF RELATIONSHIP

Though the fraction is not unknown to the pupil when he begins its systematic study, the idea of the fraction that he has developed is a very inadequate one. He says, for instance, the fractional form of expression in speaking of "half a day," he does not necessarily think of the half day as the relation to the whole day. The half day is not considered by him as a division of a stated unit of time, it is thought of as a unit of time by itself, just as he would think of an hour, or of thirty minutes as a unit of time. When he buys a half pint of cream from the store, though he may bring at the same time a pint of milk, he does not think of the half pint as a division of the pint. The half pint of cream is as one bottle, just as the pint of milk is in one bottle. There is an apparent relation between the two so that each is considered as one, the other. The half-pint bottle is a whole bottle, as the same as the pint bottle or the quart bottle is a whole bottle. When the girl buys a half-yard of ribbon for her doll, she buys a whole piece of ribbon. The quarter is a whole piece of ribbon. The piece shows within itself its measure, it is a unit, not a part of a unit. (For example)

the original stick of candy is broken in two, and each piece is called "a half," the pieces given away and the piece that is retained do not necessarily improve their relation with the original stick. Each piece is a unity, and may not be considered as a part of a unity. The child's everyday experience with parts do not improve him with the characteristic features and relationship of parts. They afford him a convenient way of expression, not an adequate idea.

Describing of the idea of relation of the part to the whole is, however, improved by the exercise of finding one of the equal parts of a quantity by dividing the quantity by the number of parts desired. The answer secured, which is the quotient of the division, is not an absolute result, but one that must be thought of in its relation to the number that has been divided. The question asked at the outset, and usually repeated as part of the description of the quotient, serves to bring the quotient into proper relation with the original number. For example, the question, "What is one half of 24?" serves to describe the answer, 12, in its relation to 24. The pupil through such exercises is required to give some consideration to the fraction as an expression of relation.

These observations suggest the point that the fraction, or the idea of a part, is not a separate and distinct idea, however much it may be given an isolated use. A given fraction, one half, for example, may be spoken of, written, and used in computation as though its meaning were wrapped up entirely within itself. But 'one half' as an actual idea always refers to the thing that has been divided; and, though it often appears in actual experience as a unity within itself — as half a day, half a yard, etc. — its meaning can never be clear until it has been derived from the two quantities, or amounts, that designate the relation that one half expresses.

On the other hand, the pupil will have to learn to deal with the idea 'half' in a variety of situations that do not

permit constant reference to such quantities and amounts as give the relation that is expressed by one half. He will have to deal with one half as the same term for division with a third, a fourth, and so on. He will have an access or opportunity to be going back in his thinking to get the various relations established. When he deals with the half and the third, for example, he will need to deal with them as relations to each other, and not each in its relation to its particular set of dual quantities or dual amounts. In other words, he must learn to deal with the half as a single abstraction, and as a part of a larger abstraction; he must gain the ability to deal with the half as a separate idea, although in reality it is never a separate idea. Or, to state the matter in another way, he must learn to deal with the fraction as a separate idea, though it is always a related idea.

Since the idea of the fraction is the idea of the part (of something) — which indicates that its true meaning is one of relationship — it can never be used intelligently as an isolated expression or as an expression of an isolated amount if its meaning is neglected or abandoned. One does not need, of course, to be thinking of the meaning of the fraction while he is trying to use it; but he does need to know that its true meaning be not neglected. Some means must be at hand, therefore, to develop the meaning of the fraction, so that while it is being used as an isolated expression of quantity or amount, its true meaning of relationship will still be retained.

III. SIZE AND NUMBER

If the pupil has been engaged in the kinds of activities that have been suggested in the preceding chapters, he will have gained from his studies of groups certain ideas that will be very useful and necessary ones in his study of parts. In his study of groups he has had to deal, whether or not gently or not, with the idea of size and the idea of number. He will now, in his study of parts, have to deal, whether

intelligently or not, with the same ideas. These ideas are given expression in what are called the terms of the fraction; namely, the denominator and the numerator. If now the pupil can be led to give his attention to the size of the parts, expressed by the denominator, as well as to the number of parts, expressed by the numerator, he will have two serviceable ideas to aid him in keeping the meaning of the fraction clear. The amount of the fraction is always expressed in terms of size of parts and number of parts, and attention to these two ideas gives meaning to, and preserves meaning for, the expression of the fraction.

The idea of size is never entirely absolute, it is always in some degree a related idea. Being an idea of relation, it easily adjusts itself as a carrier of the relational meaning of the fraction. What is more to the point, however, is the fact that the individual can develop considerable familiarity with certain standard sizes or certain frequently studied sizes, whether of groups or of parts. If now the pupil can develop familiarity with certain commonly used sizes of parts, he can resort to the expression and use of a given size of part (shown by the denominator) with those of a given number of parts (shown by the numerator) to make meaningful and keep meaningful a given fractional expression that he can think of and deal with as an expression of an isolated amount. Unless the pupil does develop acquaintance with sizes of parts, he will be unable to deal intelligently with the fraction as a single abstraction, either because he has no meaning for it or because he is compelled to interrupt his thinking by the necessity of returning to concrete experiences for meaning. Ideas of size and number are the central ideas in the study of parts, just as they are the central ideas in any systematic study of groups.

IV. PURPOSE OF THE CHAPTER

The purpose of the present chapter is to suggest means of developing the idea of the fraction, so that it may be clear

and unmistakable in whatever form it may be expressed and to present and discuss the three ways of expressing and dealing with the fractional idea. The chapter will seek to emphasize the fact that the fraction retains the same characteristics and uses, whether it is expressed as a common fraction, as a decimal, or as a percent. For the sake of convenience, however, the three forms of expression will be considered in the order (1) common fractions, (2) decimal fractions, and (3) percentage.

V. Common Fractions

1. Studying Separate Parts

Before the pupil can undertake the systematic comparison of fractions or begin the use of fractional words as adjectives, on the other, he must be led to develop a fairly definite notion of each fraction as a separate entity. He should have brought to his attention, as objectively and as favorably as possible, the idea of division, the equality of the parts considered in the division, the fact that the parts considered make the original whole, how the size of the parts is determined and how a given number of parts is indicated. The half, the third, the fourth, etc., are studied much by stating that half as one of the two equal parts of the thing considered, the third as one of the three equal parts, the fourth as one of the four equal parts, and so on. Objective demonstration is necessary in order to impress the essential characteristics and uses of the fractions to be studied.



Let us suppose the fourth is being studied. Any object that may be easily divided into fourths will be equally suitable to use. Let us say that a circle is chosen. Let the circle be divided into four equal parts. Next, the equality of the

parts is again called to attention by questioning, discussion, etc. What each part is called (*one-fourth*) is emphasized. The number of fourths in the whole circle is impressed. Finally, how the parts are expressed in writing is pointed out.

From the beginning, the figure above the line of the fraction must be emphasized as the one that shows the number of parts: thus, one fourth is written, $\frac{1}{4}$, two fourths, $\frac{2}{4}$, etc. Since the number of parts (fourths) in the whole circle is shown thus, $\frac{1}{4}$, it should be made clear that the 4 above the line shows all the parts (fourths) in the circle, that is, the number of parts into which the circle was divided. Likewise, from the beginning, the figure below the line must be emphasized as the one that shows size. In this case, though written as 4, it does not show four, nor is it read as four, but as *fourths*. Since the idea of size is relative, there should perhaps be some comparison of sizes from the beginning, however inexact the comparison may have to be. Thus, while the pupils are studying fourths, they may be led to notice comparisons with thirds and halves, which have already been studied.



Which is the largest -- one half, one third, or one fourth? Which is the smallest? Which is the larger -- one third or one fourth? Which is the smaller? The purpose of such questions is to contribute to the developing idea of size and to develop the fact that the larger the figure in the denominator, the smaller is the size of the part.

It is not to be expected that the pupil will gain a complete idea of size at the very beginning. The idea of size, through the comparison of sizes, must, however, be called to attention from the beginning. Further reference to the idea of size of parts will be made presently.

2. Division and Comparison of Parts

The work indicated in the foregoing chapter paragraphs would the pupil have gained fairly distinct ideas of each of the most commonly used common fractions. The next step is to enlarge his idea of each of the fractions by the method of analysis and synthesis. Such a method leads to a comparison of a given fraction with those most closely related to it and prepares for the steps of study that must necessarily follow. The method may be illustrated, as follows:

Illustration Lesson on Stripping Hairs

Let us divide a circle into 2 equal parts. Each part is $\frac{1}{2}$ of the circle. There is an upper $\frac{1}{2}$ and a lower $\frac{1}{2}$. Let us summarize how much of the whole circle $\frac{1}{2}$ is. Now let us divide the same circle



into 4 equal parts. We can do that by dividing each half into 2 equal parts. We will keep the heavy line to show the circle as it was first divided into halves, and use lighter lines to show it divided into fourths. Our circle will now look like this one:



There are four fourths ($\frac{1}{4}$) in the whole circle.

Count the fourths in $\frac{1}{2}$ of the circle. You can see that in one half ($\frac{1}{2}$) of the circle there are two fourths ($\frac{1}{4}$). We can say of this, "One half equals two fourths," and we can write it like this, $\frac{1}{2} = \frac{2}{4}$.

Now let us divide the circle into 8 equal parts. We can do this by dividing each fourth into 2 equal parts. We will keep the heavy line to show the circle as it was first divided into halves,

and use lighted lines to show it divided into eighths. Our circle will now look like this one:



There are eight eighths ($\frac{1}{8}$) in the whole circle.

Count the eighths in $\frac{1}{2}$ of the circle. You can see that there are four eighths ($\frac{4}{8}$) in one half ($\frac{1}{2}$) of the circle. We can say it like this, "One half equals four eighths," and we can write it like this, $\frac{1}{2} = \frac{4}{8}$.

In the same way we can show other fractions that $\frac{1}{2}$ equals:



- | | |
|----------------------------------|---|
| Circle 1 shows $\frac{1}{2} = 1$ | How many fourths in $\frac{1}{2}$? $\frac{1}{2} = 2$? |
| Circle 2 shows $\frac{1}{2} = 2$ | How many eighths in $\frac{1}{2}$? $\frac{1}{2} = 4$? |
| Circle 3 shows $\frac{1}{2} = 3$ | How many eighths in $\frac{1}{2}$? $\frac{1}{2} = 6$? |
| Circle 4 shows $\frac{1}{2} = 4$ | How many tenths in $\frac{1}{2}$? $\frac{1}{2} = 5$? |
| Circle 5 shows $\frac{1}{2} = 5$ | How many twentieths in $\frac{1}{2}$? $\frac{1}{2} = 10$? |

In the same way with other circles we could show

- | | |
|-------------------|---|
| $\frac{1}{4} = 1$ | How many fourteenths in $\frac{1}{4}$? $\frac{1}{4} = 7$? |
| $\frac{1}{4} = 2$ | How many sixteenths in $\frac{1}{4}$? $\frac{1}{4} = 8$? |
| $\frac{1}{4} = 3$ | How many eighteenthths in $\frac{1}{4}$? $\frac{1}{4} = 9$? |
| $\frac{1}{4} = 4$ | How many twentiethths in $\frac{1}{4}$? $\frac{1}{4} = 10$? |

In similar manner, the third and the fourth may be studied. When the pupils have learned to work out the relations indicated through actual divisions of the parts being studied, and have summarized their work in a table of equivalents, like the following:

$$\begin{array}{l}
 \frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} = \frac{7}{14} = \frac{8}{16} = \frac{9}{18} = \frac{10}{20}, \text{ etc.} \\
 \frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12} = \frac{5}{15} = \frac{6}{18}, \text{ etc.} \\
 \frac{1}{4} = \frac{2}{8} = \frac{3}{12} = \frac{4}{16} = \frac{5}{20}, \text{ etc.}
 \end{array}$$

they should be shown a shorter method of determining equivalents. The shorter method may be illustrated as follows:

Explaining Partial Cancellation

Let us try another way of finding out about fractions. The way we have been using of dividing rather (or multiplying when the matter) into parts and counting the parts is a very good way. The only thing the matter with that way is that it is very difficult for us to divide anything into more than eight or ten equal parts and to do it very exactly. The more parts we try to divide a thing into, the more tedious the task becomes. Let us try a shorter, easier way. This way we will call *partial cancellation*.

Cancellation means striking out by dividing.

In using the method of *partial cancellation* we need to remember that 1 is the same as $\frac{1}{1}$, or $\frac{1}{2}$, or $\frac{1}{3}$, or $\frac{1}{4}$, etc.

Suppose we were asked

(in words) "One half as large as twelve" or
(in writing) $\frac{1}{2} \times 12$?

We could find the answer by thinking like this,

If 1 equals 10 tenths ($\frac{10}{10}$),

$\frac{1}{2}$ equals $\frac{1}{2}$ of 10 tenths, or five tenths ($\frac{5}{10}$).

We can do our thinking to ourselves and make what we think like this,

$$\begin{array}{l} \frac{1}{2} \text{ of } \frac{10}{10} = 5 \\ \frac{1}{2} = 5? \quad \frac{1}{2} = 5? \\ \frac{1}{2} \text{ of } \frac{10}{10} = 5 \quad \frac{1}{2} \text{ of } \frac{10}{10} = 5 \end{array}$$

This is a short way to help us in thinking. If 1 equals eighteen eighths, one third equals one third of eighteen eighths or six eighths, and two thirds equal two thirds of eighteen eighths or twelve eighths.

We can turn back to the circles we have been using in the other circles or any other objects, and count the tenths or the eighths in $\frac{1}{2}$, and the eighths in $\frac{2}{3}$, and so on, find that our answers are correct.

Notice how the answers to these questions are found in the shorter, easier way of partial cancellation.

$$1. \frac{3}{4} = \frac{3}{4}?$$

$$2. \frac{1}{2} = \frac{1}{2}?$$

$$1. \frac{3}{4} \text{ of } \frac{3}{4} = \frac{3}{4}$$

$$2. \frac{1}{2} \text{ of } \frac{1}{2} = \frac{1}{4}$$

$$3. \frac{3}{4} = \frac{1}{2}?$$

$$4. \frac{1}{2} = \frac{1}{2}?$$

$$3. \frac{3}{4} \text{ of } \frac{1}{2} = \frac{3}{8}$$

$$4. \frac{1}{2} \text{ of } \frac{1}{2} = \frac{1}{4}$$

3 Comparing Parts

When the pupils have made and studied such comparisons as have been indicated in the method of analysis and synthesis, they should be made conscious of what things are involved in a comparison, in order that they may be able to compare any fraction with any other one. The idea of size of parts and the idea of number of parts must be made as impressive as possible. The following paragraphs indicate the type of lessons that may lead the pupil to give his attention to both size and number of parts when he undertakes comparisons.

Illustrative Lessons on Comparing Parts

A good way to learn about fractions is to compare them. When a person compares two things, he is able to find out something new about both of them. He is able to tell whether they are both the same size or one is larger than the other. If they are of different sizes, he can tell which is larger and which is smaller, and how much larger and how much smaller. Last year you learned some things about numbers by comparing them. For example, in comparing 8 and 12, you learned that 8 is 4 less than 12, and that 12 is 4 more than 8. Now you will have a chance to learn some new things about halves, thirds, fourths, fifths, and so on, by comparing them. You will compare one half and one third, for example, and you will find out which is the larger and which is the smaller, and, what is more important, you will learn *exactly* how much larger and how much smaller. When you have learned to make

comparisons between a number of fractions goes with and we know a lot more about them than goes the way that goes and we be better able to use them.

In making a comparison, a person has to pay attention to *both* things. If he pays attention to either one of them only and forgets the other one, he makes a great mistake. Let us see if you can tell what these two things are. You will have to know them and always remember them to compare fractions.

1. Henry and James were comparing the amounts of money they had saved. Henry had several coins and James had several coins. Who had the more money, Henry or James? (I tell why you cannot solve this problem.)

2. Susan had 5 coins and Margaret had 30 coins. Who had the more money, Susan or Margaret? (I tell why you cannot solve this problem.)

3. Mary had some dimes in her purse and John had some dimes in his. Who had the more money, Mary or John? (I tell why you cannot solve this problem.)

4. Robert had 7 nickels and Walter had 4 dimes. Who had the more money, Robert or Walter? (You can solve this problem. Tell why you can.)

Let us study these four problems. Problem 1 does not tell us either the number or the size of the coins each had. Therefore, I am not much better. It tells us the number of coins each had but it forgets to tell us the size. Number without size does not help us very much. Problem 3 is just the opposite. It tells us the size of the coins (dimes), but it forgets to tell us the number each had. Size without number does not help us very much. Let us go to Problem 4. It tells us everything we would be known. It tells us both the number and the size of the coins each had. We can compare the money Robert had with the money Walter had. Because we know the number and the size of the coins.

In order to compare, we must know and pay attention to the number and size.

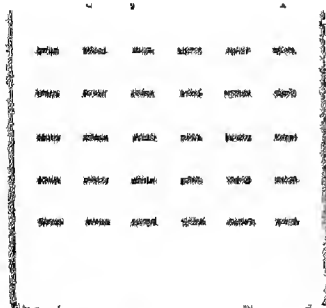
Notice how we work Problem 4. We do not compare the 7 nickels with the 4 dimes, and say that Robert had the more money because he had 7 coins and Walter had only 4. The number and the size are not of the same size, so we put them in two groups of the same size: 7 nickels = 35¢, and 4 dimes = 40¢. We can

compare 36 with 42, because the two arrays are built thought of as rows of the same size. That is, both are six wide.

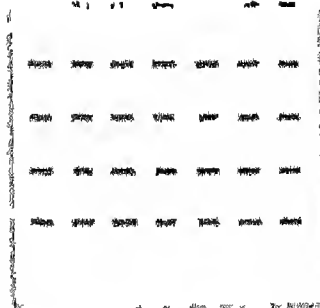
Herb said to Jane, "Which would you rather have — 3 rows of nearly or 5 rows of nearly?" Jane answered, "It depends on the size of the rows."

Which would you rather have — 1 dish of ice cream or 2 dishes of ice cream?

§ In the first grade and second-grade rooms at our school they have tables arranged as rows. In the first-grade room there are 6



First Grade



Second Grade

rows with 6 tables in a row. In the second-grade room there are 5 rows with 7 tables in a row. In which room are there the more tables?

This problem is easy to solve, because we are told both the number of rows and the size of each row in each of the rooms. But notice — we do not try to compare 6 rows in one room with 5 rows in the other or the size of a row (6 tables) in one room with the size of a row (7 tables) in the other. It would be foolish for the first-grade pupils to say, "We have more tables, because we have 6 rows, and you have only 5 rows," and it would be foolish for the second-grade pupils to say, "We have more than you, because we have 7 tables in each of our rows." Instead of trying to make a foolish comparison like that, we just think the number and the size of the rows in the two rooms into *one*; that is, into something of the same size. We just think, "Six sixes are thirty-six," and "Five sevens are thirty-five," and compare the 36 with the 35.

In order to compare, we must (1) know and pay attention to both *number* and *size*, and (2) think the things that are of different sizes into *things of the same size*.

Now let us try to compare two fractions $\frac{1}{2}$ and $\frac{1}{3}$. Just before we begin, let us review what a fraction means. A fraction has two letters. The one above the line is the numerator and the one below the line is the denominator. The numerator of a fraction tells the number of parts and the denominator tells the size of each part. We, in comparing two fractions like $\frac{1}{2}$ and $\frac{1}{3}$, we always have given the two things we cannot tell at first your attention to, number and size. All we need to do is to find the fractions that are of different sizes but have the same size.

It will be found in comparing $\frac{1}{2}$ and $\frac{1}{3}$ that the numbers of the parts are different. It is true that $\frac{1}{2}$ and $\frac{1}{3}$ are not about the different size. And it seems it will be impossible to try to compare the size. The best way is to try to forget all about the numbers. There is only one thing to do that is to think the two fractions $\frac{1}{2}$ and $\frac{1}{3}$ as the fractions of the same size. When we get these two fractions in the same size, we can easily compare the number of parts. The size of the number of parts in the other.

Which would you rather have? A stick of candy or 2 sticks of candy? That depends on the size of the stick. If it is small, 2 is better.

Which is the larger? A half or 2 thirds? It depends on the size of the parts.

Let us compare $\frac{1}{2}$ and $\frac{1}{3}$ by using circles. To compare them we must either actually draw them or compare them. We can draw one circle into 2 equal parts and then draw another circle into 3 equal parts.



We can tell that one of the halves is $\frac{1}{2}$ and the other is $\frac{1}{2}$, but we cannot tell exactly what $\frac{1}{3}$ is. We can only look at it. In order to compare the two $\frac{1}{2}$ and $\frac{1}{3}$, we must cut the half and the thirds into parts of the same size. We can do that by cutting each circle into 6 equal parts.



noticing how many parts there are in $\frac{1}{4}$ and how many parts there are in $\frac{3}{6}$.

We now can compare the $\frac{1}{4}$ and the $\frac{3}{6}$, because we have each fraction made into parts of the same size. In $\frac{1}{4}$ there are 3, and in $\frac{3}{6}$ there are 3. We can see that $\frac{1}{4}$ is $\frac{1}{6}$ less than $\frac{3}{6}$, or that the fraction $\frac{1}{4}$ is $\frac{1}{6}$ more than the fraction $\frac{2}{6}$.

Thinking the Comparison

In comparing fractions, like $\frac{1}{4}$ and $\frac{3}{6}$, we can use circles or any other objects, and actually change the $\frac{1}{4}$ to $\frac{3}{6}$ and the $\frac{3}{6}$ to $\frac{1}{4}$ by cutting each into parts. Instead of using objects to change the fractions actually, we can think them into fractions of the same size. This is the shorter, easier way. Let us see.

We decide to change $\frac{1}{4}$ into sixths and $\frac{3}{6}$ into sixths, so that we can compare them. We change both by the method of partial cancellation, remembering in each case that there are 6 in 1.

$$\begin{array}{ll} \frac{1}{4} \text{ of } \frac{3}{6} = \frac{3}{24} & \text{We can now compare } \frac{1}{4} \text{ and } \frac{3}{6}, \text{ because we have} \\ \frac{2}{6} \text{ of } \frac{3}{6} = \frac{6}{24} & \text{changed both into parts of the same size, into} \\ & \text{sixths.} \end{array}$$

Sometimes in comparing two fractions, like $\frac{1}{4}$ and $\frac{3}{6}$, in order to make them into parts of the same size, we have to change both fractions. Sometimes we need to change only one of the fractions. Let us compare $\frac{2}{3}$ and $\frac{1}{4}$.

We cannot compare $\frac{2}{3}$ and $\frac{1}{4}$ as they are, because the parts, thirds and fourths, are not of the same size. We know, however, that we can change the $\frac{2}{3}$ to sixths, and the $\frac{1}{4}$ is already in sixths without being changed. So we just change the one fraction to sixths, $\frac{2}{3}$ of $\frac{2}{3} = \frac{4}{6}$, and compare it with the $\frac{1}{4}$.

Let us compare $\frac{1}{4}$ and $\frac{1}{6}$. Since we can change fourths to

Find by the integer decomposition set out on p. 379, and see whether it can be divided even (that is, without a remainder), by the other decomposition, which is 4. Since 4 cannot be divided even by 4, multiply 4 by 2, which gives 8, and see whether 12 can be divided even by 4. Since 12 can be divided even by 4, write 12 in the common denominator, and we change both $\frac{1}{2}$ and $\frac{1}{3}$ to twelfths by partial cancellation.

$$\frac{1}{2} \text{ of } \frac{1}{3} = \frac{1}{6}$$

$$\frac{1}{3} \text{ of } \frac{1}{2} = \frac{1}{6}$$

Let us compare $\frac{1}{2}$ and $\frac{1}{3}$. Since 3 cannot be divided even by 2, we multiply 3 by 2, which gives 6. But 6 cannot be divided even by 3, so we next multiply 3 by 3, which gives 9. Since 9 can be divided even by 3, write 9 in the common denominator, and we change both $\frac{1}{2}$ and $\frac{1}{3}$ to twenty-fourths by partial cancellation.

$$\frac{1}{2} \text{ of } \frac{1}{3} = \frac{1}{6}$$

$$\frac{1}{3} \text{ of } \frac{1}{2} = \frac{1}{6}$$

Let us compare $\frac{1}{2}$ and $\frac{1}{3}$. 12 cannot be divided even by 12; so we multiply 12 by 2 which gives 24. 24 cannot be divided even by 12 so we multiply 12 by 3, which gives 36. 36 cannot be divided even by 12, so we multiply 12 by 4, which gives 48. Since 48 can be divided even by 12, we know that 48 is the common denominator, and we change $\frac{1}{2}$ and $\frac{1}{3}$ to twenty-fourths by partial cancellation.

$$\frac{1}{2} \text{ of } \frac{1}{3} = \frac{1}{6}$$

$$\frac{1}{3} \text{ of } \frac{1}{2} = \frac{1}{6}$$

To find the common denominator in comparing two fractions, first see if the larger denominator can be divided even (that is, without remainder) by the smaller. If it cannot, multiply it (the larger denominator) in turn by 2, 3, 4, etc., until you get a number that can be divided even.

When you know what the common denominator is, change the fractions by the method of partial cancellation.

5. Addition and Subtraction

The addition and subtraction of fractions should be delayed until the pupil has enlarged and clarified his ideas of the most frequently used common fractions. The reason

for this delay is that we will not be prepared to add or to subtract fractions with intelligence until we can add the numerators of the fractions to be dealt with, or the denominators to the other. Just as previous discussions have pointed out that there is a great gap between counting and the consideration of groups (which ought to be filled with the consideration of groups through enumeration and through analysis and synthesis of groups) so our present discussion has been intended to indicate that a somewhat similar gap exists between the viewing of each fraction by itself as one or more of the equal parts of something and the consideration of parts by addition and subtraction which involves the dealing with somewhat similar systematic studies of groups. Paying attention to the size and number of groups which are involved is viewed in its relation to number processes for addition and subtraction. Addition and subtraction give distinct opportunity to study size and number of groups.

The methods by which size and number of groups that are emphasized have been illustrated in connection with preceding topics. Such illustrations would not be surprising here. We should bear in mind, however, that the addition and the subtraction of fractions introduce what appear at first glance to be new operations, but that they are not new operations that in every case give opportunity for constant reference to previous studies of division of groups into equivalents, of comparisons, and of other mental operations. Let us mention the outstanding ones of these from previous work.

Reduction is one of the new operations which the student is to an addition or to a subtraction to use. "Let us reduce," $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, for example. At the present time reduction should be made through an investigation of equivalent fractions of which has previously been presented. $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$, etc.; $\frac{1}{3} = \frac{2}{6} = \frac{4}{12}$, etc. The reduction may be made through a representation of a circle, or a circle originally divided into various equal parts. If the circle is divided into eight equal parts, then the fraction $\frac{1}{2}$ is represented by four of the parts.

remind himself that $\frac{2}{2}$ is the same as $\frac{1}{1}$. A number of fractions that may be reduced should be studied in a similar way and their reduced equivalents noted, written down, and compared with the originals. Finally, the general rule for reduction should be reached, namely, that both terms of the fraction may be divided by the same number without changing its value.

Other "new" processes may be introduced as follows:

Teaching More about Addition

1. Frances had a candy bar that was divided into six parts. She ate $\frac{1}{2}$ of the bar at lunch and the other $\frac{1}{2}$ at recess. How much of the bar did she eat?

Of course, this problem is too easy. You know at once that Frances ate all of her candy bar. Let us see how to write what we do in working the problem. We add $\frac{1}{2}$ and $\frac{1}{2}$, using either form.

$$\frac{1}{2} + \frac{1}{2} = \frac{2}{2} \text{ or } \frac{1}{1} = 1$$

The answer, $\frac{2}{2}$, tells us that she ate the whole bar, because we know that $\frac{2}{2} = 1$. Let us see how we can tell.

Last year you learned that a fraction means to divide, and this year you have learned that the denominator tells the size of the parts, because it shows into how many parts a thing is divided. A fraction, like $\frac{1}{2}$, or $\frac{1}{4}$, or $\frac{1}{8}$, or $\frac{1}{16}$, or $\frac{1}{32}$, means, along with other things, that the number above the line is to be divided by the number below the line. In the case of fractions, like $\frac{2}{2}$, $\frac{3}{3}$, etc., we let them stand as they are, $\frac{2}{2}$, $\frac{3}{3}$, etc., since we have not yet learned how to divide a smaller number by a larger number. We shall learn how to do that next year.

In the case of fractions, like $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, we can divide, like this:

$$\frac{1}{2} = 6 \overline{)6}$$

$$\frac{1}{3} = 3 \overline{)12}$$

John had $1\frac{1}{2}$ yards of goods. He used $\frac{1}{2}$ of a yard to make a dress for his daughter. How much of the goods had he left?

1 $\frac{1}{2}$ Since one yard is $\frac{2}{2}$ from $\frac{1}{2}$, we think the 1 into eighths ($1 = \frac{8}{8}$) and count it with the $\frac{1}{2}$, which gives $\frac{4}{8}$. We now subtract $\frac{4}{8}$ from $\frac{8}{8}$, which gives $\frac{4}{8}$, or $\frac{1}{2}$ yard.

Ann had $1\frac{1}{2}$ yards of tape. She used $\frac{1}{2}$ of a yard to tape the handles of her tennis racket. How much tape had she left?

1 $\frac{1}{2}$ We first change the $\frac{1}{2}$ and the $\frac{1}{2}$ to parts of the same size: $\frac{1}{2}$ of $\frac{2}{2} = \frac{1}{1}$, $\frac{1}{2}$ of $\frac{2}{2} = \frac{1}{1}$. Since $\frac{1}{1}$ is larger than $\frac{1}{1}$, we subtract, and get $\frac{1}{1}$, and as we have nothing to take from 1, we leave 1 left when the answer is $1\frac{1}{2}$ yards.

1 $\frac{1}{2}$ George had $1\frac{1}{2}$ yards of tape. He used $\frac{1}{2}$ of a yard to tape the handles of his tennis racket. How much tape had he left?

1 $\frac{1}{2}$ We first change the $\frac{1}{2}$ and $\frac{1}{2}$ to parts of the same size: $\frac{1}{2}$ and $\frac{1}{2}$. Since $\frac{1}{2}$ is smaller than $\frac{1}{2}$, we change the 1 to tenths ($1 = \frac{10}{10}$), and count it with the $\frac{1}{2}$, which gives $\frac{5}{10}$. We now subtract $\frac{5}{10}$ from $\frac{5}{10}$, which gives $\frac{0}{10}$ yard. That answer is $\frac{0}{10}$ yard.

Things to remember and do

1. See that the fractions are in parts of the same size.
2. Notice which fraction is the larger.
3. When necessary, change the whole number in the minuend to a fraction, with parts of the same size, and count it with the fraction in the minuend.

Teaching Carrying in the Addition of Mixed Numbers

If you remember everything you have learned about fractions, you are now ready to learn how to add whole numbers and fractions to whole numbers and fractions. Sometimes you will have to carry; sometimes you will not. Let us see.

1. Add $3\frac{1}{2}$ and $4\frac{1}{2}$

3 $\frac{1}{2}$ $\frac{1}{2}$ of $\frac{2}{2} = \frac{1}{1}$ First, change the fractions to parts of the same size, and add $\frac{1}{2} + \frac{1}{2} = \frac{2}{2}$, which is less than 1. Write $\frac{2}{2}$ and add the whole numbers. The answer is $7\frac{2}{2}$.

4 $\frac{1}{2}$ $\frac{1}{2}$ of $\frac{2}{2} = \frac{1}{1}$

7 $\frac{2}{2}$

2. Add $3\frac{1}{2}$ and $4\frac{1}{2}$.

$$\begin{array}{r} 3\frac{1}{2} \\ 4\frac{1}{2} \\ \hline 8\frac{2}{2} \end{array}$$

First, change the fractions to parts of the same size and add $\frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$. Then add carrying 1 to the whole numbers and add with them. The answer is $8\frac{2}{2}$.

Teaching Carrying in the Subtraction of Mixed Numbers

If you remember everything you have learned about fractions, you are now ready to learn how to subtract mixed numbers and fractions from whole numbers and fractions. In subtraction you will have to carry, sometimes you will not. Let us see.

1. Subtract $2\frac{1}{2}$ from $4\frac{1}{2}$

$$\begin{array}{r} 4\frac{1}{2} \\ 2\frac{1}{2} \\ \hline 2\frac{0}{2} \end{array}$$

First, change the fractions to parts of the same size. Then the upper fraction is $\frac{1}{2}$ and the lower fraction is $\frac{1}{2}$. Subtract $\frac{1}{2} - \frac{1}{2} = 0$. Write 0 over the whole numbers. The answer is $2\frac{0}{2}$.

2. Subtract $5\frac{1}{2}$ from $8\frac{1}{2}$

$$\begin{array}{r} 8\frac{1}{2} \\ 5\frac{1}{2} \\ \hline 3\frac{0}{2} \end{array}$$

First, change the fractions to parts of the same size. Then, the upper fraction is $\frac{1}{2}$ and the lower fraction is $\frac{1}{2}$. Subtract $\frac{1}{2} - \frac{1}{2} = 0$. Write 0 over the whole numbers. The answer is $3\frac{0}{2}$.

3. Subtract $5\frac{1}{2}$ from 8

$$\begin{array}{r} 8 \\ 5\frac{1}{2} \\ \hline 2\frac{1}{2} \end{array}$$

Use 1 (changed to fifths in the statement) subtracted $\frac{1}{2}$ from 5. Write $\frac{1}{2}$. Since one has been used, seven are left. Subtract 5 from 7. The answer is $2\frac{1}{2}$.

6 Multiplying Fractions

The order of presentation of the exercises follows the multiplication of fractions is, (1) a fraction is multiplied by a

whole number (2) a whole number multiplied by a fraction, (3) a fraction multiplied by a fraction, (4) a whole number multiplied by a mixed number, and the reverse, and (5) a mixed number multiplied by a mixed number.

In taking up formal work in the multiplication of fractions, there is saved by presenting first the repetition of a fraction a given number of times, $4 \times \frac{1}{2}$, or the multiplication of a fraction by a whole number.¹

Each step should be presented with as much of objective demonstration as is necessary for the pupils to understand what is required in the multiplication and how to proceed. When they understand, they should be provided with plenty of practice in carrying through the given step before proceeding to the next. Illustrations will be given of the first and third steps.

Teaching How to Multiply a Fraction by a Whole Number

1. How many grape fruits will it take to serve 6 persons if each person gets $\frac{1}{2}$ of a grape fruit?

It will take as many halves as this for all:



The problem can be worked either way, as follows:

$$\begin{array}{r} \frac{1}{2} \\ 6 \\ \hline 3 \end{array} \quad \text{or} \quad 6 \times \frac{1}{2} = \frac{6}{2} = 3$$

2. Susan needs 4 strips of edging to go round a pillow she is making. Each strip must be $\frac{3}{4}$ yard long. How many yards of edging does she need?

How much she will need can be shown by a diagram, like this:

¹ Myrtle Collier. "Learning to multiply fractions." *School Science and Mathematics*, 22: April, 1922, 324-329, esp. 326.

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The 4 strips would make $2\frac{3}{4}$ yards

The problem can be worked either way as follows:

$$\begin{array}{r} \frac{3}{4} \\ \times 4 \\ \hline 3 \\ 12 \\ \hline 12 \end{array} \quad \text{or} \quad \begin{array}{r} 3\frac{3}{4} \\ \times 4 \\ \hline 15 \\ 12 \\ \hline 15 \end{array}$$

3. Multiply $\frac{1}{4}$ by 12

Notice the two methods. Which is the easier?

$$\begin{array}{r} A \\ 12 \times \frac{1}{4} = \frac{12}{4} = 3 \end{array} \quad \begin{array}{r} B \\ 12 \times \frac{1}{4} = 3 \end{array}$$

In Method A, we first multiply 12 by 12, then divide by 4

In Method B we first divide 12 by 4, then multiply by 1

4. Multiply $\frac{1}{5}$ by 25

$$\begin{array}{r} A \\ 25 \times \frac{1}{5} = \frac{25}{5} = 5 \end{array} \quad \begin{array}{r} B \\ 25 \times \frac{1}{5} = 5 \end{array}$$

5. Multiply $\frac{1}{7}$ by 14.

$$\begin{array}{r} A \\ 14 \times \frac{1}{7} = \frac{14}{7} = 2 \end{array} \quad \begin{array}{r} B \\ 14 \times \frac{1}{7} = 2 \end{array}$$

Method B is the easier when the whole number can be divided even by the denominator. Method A is the easier when the whole number cannot be divided even.

Rule In multiplying a fraction by a whole number, first notice if the whole number can be divided even by the denominator. If it is exact then multiply the numerator by the whole number, and divide by the denominator. If it is even, then divide the whole number by the denominator, and multiply by the numerator.

Teaching How to Multiply a Fraction by a Fraction

1. How much is $\frac{1}{2} \times \frac{1}{2}$? (by how much is $\frac{1}{2}$ of $\frac{1}{2}$?)

Let us use a diagram. Suppose we draw a line and divide it into 12 equal parts, as follows:



From A to D is $\frac{1}{3}$ of the line

From A to M is $\frac{1}{2}$ of the line

(Let us note $\frac{1}{2}$ of $\frac{1}{3}$ of the line, or $\frac{1}{6}$ of the part A to M.)

From A to E is $\frac{1}{3}$ of the part A to M

From A to I is $\frac{1}{2}$ of the part A to M

Counting from A to I we have $\frac{1}{6}$ of the whole line.

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}, \text{ or } \frac{1}{2} \text{ of } \frac{1}{3} = \frac{1}{6} \quad \left(\begin{array}{l} \text{Note } 2 \text{ 4's are 8} \\ 3 \text{ 5's are 15} \end{array} \right)$$

In multiplying a fraction by a fraction, multiply the numerators, then multiply the denominators.

2. Multiply, $\frac{2}{3} \times \frac{1}{2}$

$$\frac{2}{3} \times \frac{1}{2} = \frac{2 \times 1}{3 \times 2} = \frac{2}{6} = \frac{1}{3} \quad \text{or} \quad \frac{1}{3} \times \frac{2}{2} = \frac{1}{3}$$

Method B is the method of cancelling. 2 will divide the 2 in the numerator of one fraction and the 4 in the denominator of the other, and 3 will divide the 3 in the numerator of one fraction and the 3 in the denominator of the other.

$$3. \text{ Multiply, } \frac{2}{3} \times \frac{1}{2} = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

$$4. \text{ Multiply, } \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

In multiplying a fraction by a fraction, the student can be led to see that he can divide the numerators and the denominators by the same numbers as much as possible first, multiplying the numerators and then multiplying the denominators.

In the multiplication of mixed numbers which are used as numbers (listed as a fourth phase of the multiplication of fractions) the pupil is given further practice in what he has already learned, and at the same time he is prepared to change his work about him in order to realize the fact that he can use two or more partial products that result in one.

$$\begin{array}{r}
 24 \\
 5\frac{1}{2} \\
 \hline
 16 \\
 120 \\
 120 \\
 \hline
 37
 \end{array}
 \qquad
 \begin{array}{r}
 24\frac{1}{2} \\
 12 \\
 12 \\
 \hline
 37\frac{1}{2}
 \end{array}$$

In the multiplication of mixed numbers (listed as a fifth phase) the pupil is given additional practice in multiplying fractions, in the changing of mixed numbers to improper fractions, and in the changing of improper fractions to mixed numbers.

$$2\frac{1}{2} \times 4\frac{1}{2} = 10\frac{1}{4} + 2\frac{1}{2} = 12\frac{3}{4}$$

The method of changing a mixed number to an improper fraction should be demonstrated separately, as in the following example:

Teaching How to Change a Mixed Number to a Fraction

1. Suppose we want to change $2\frac{1}{2}$ to a fraction. That is, find how many fourths in $2\frac{1}{2}$.

We know that in 1 there are four fourths, $1 = \frac{4}{4}$.

Then in 2 there are eight fourths, $2 = \frac{8}{4}$.

$\frac{8}{4}$ and $\frac{1}{2}$ are $\frac{9}{4}$.

We can change $2\frac{1}{2}$ to fourths more quickly by thinking: "Two eights are eight and one is nine fourths," $\frac{9}{4}$.



2. Let us change $1\frac{1}{2}$ to a fraction.

We think, "Eight halves are twenty-four and five are twenty-nine halves," $4\frac{5}{2}$.

7 Dividing Fractions

Since the method of dividing by a fraction is the same as the method of multiplication, except that the divisor is inverted, it follows that the pupil should first be well acquainted with the process of multiplication and then should be made to understand the reason why the inversion of the divisor is necessary. To accomplish the latter, he must review the meaning of division and keep the meaning constantly before him as he proceeds to divide.

The following sample lessons are included in order to illustrate the means of reviewing the meaning of division as a prelude to division.

Teaching the Dividing of Fractions

After learning to multiply fractions, you will find it very easy to divide fractions. The rule in dividing by a fraction is easy to remember and to use. *In dividing by a fraction, invert the divisor and multiply.*

Invert means to turn over.

When $\frac{1}{2}$ is inverted, it becomes $\frac{2}{1}$.

When $\frac{2}{3}$ is inverted, it becomes $\frac{3}{2}$.

When $\frac{3}{4}$ is inverted, it becomes $\frac{4}{3}$.

When $\frac{4}{5}$ is inverted, it becomes $\frac{5}{4}$, or 2.

When $\frac{5}{6}$ is inverted, it becomes $\frac{6}{5}$, or 3.

When $\frac{1}{2}$ is inverted, it becomes $\frac{2}{1}$

When $\frac{1}{3}$ is inverted, it becomes $\frac{3}{1}$

Let us see why, when dividing by a fraction, we multiply the dividend and multiply. We will first review what division means.

In the second and third grades you learned that division always asks a question. You learned that

2)6 asks, "How many times is six?"

3)15 asks, "How many threes in fifteen?"

4)20 asks, "How many fours in twenty?"

17)850 asks, "How many seventeens in eight hundred fifty?"

In dividing by a fraction, we use a different form of asking the question, but the kind of question asked is exactly the same. In dividing 5 by $\frac{1}{4}$, for example, instead of asking how many $\frac{1}{4}$'s in 5, to ask the question, we use this form, $5 \div \frac{1}{4}$

$6 \div 2$ asks the same question as $2 \div 6$

$15 \div 3$ asks the same question as $3 \div 15$

$60 \div 4$ asks the same question as $4 \div 60$

$850 \div 17$ asks the same question as $17 \div 850$

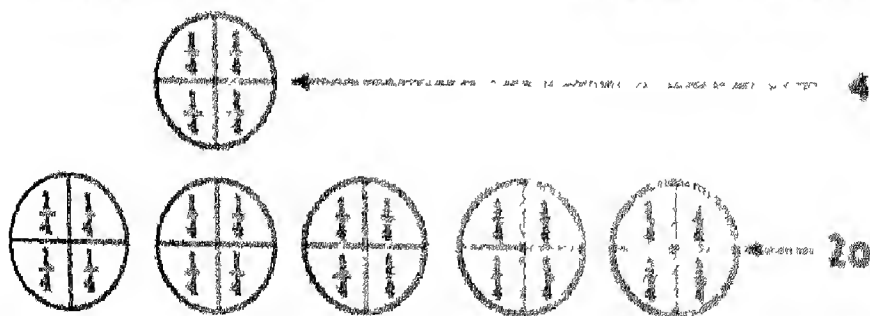
$5 \div \frac{1}{4}$ asks the same question as $\frac{1}{4} \div 5$

What question does $5 \div \frac{1}{4}$ ask?

Let us see how to find the answer.

NUMBER OF OBJECTS

SIZE OF EACH OBJECT



In order to find how many fourths there are in 5, we find out ourselves how many fourths there are in 1.

Since there are 4 fourths in 1, in 5 there are as many fourths as 5×4 , or 20.

$$3 \div \frac{1}{2}$$

$3 \div \frac{1}{2} = 20$ (We can write the answer $\frac{20}{1}$, but we know that $20 = 1 = 20$.)

1. Helen uses $\frac{1}{2}$ of a yard of crepe paper to make a paper doll. How many dolls can she make with 3 yards of crepe paper?

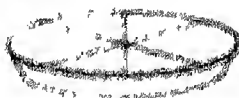
$$3 \div \frac{1}{2} = \quad \quad \quad 3 \times 2 = 15$$

$$2. \text{ Find, } 7 \div \frac{1}{2} \quad \quad \quad 7 \div \frac{1}{2} = \quad \quad \quad 7 \times 2 = 14$$

Teaching the Dividing of a Fraction by a Fraction

The rule is the same. In dividing by a fraction, invert the divisor and multiply.

Mother has $\frac{1}{2}$ of a pie. How many people can she serve if she gives each person $\frac{1}{4}$ of a pie?



The real question is, "How many sixths are there in $\frac{1}{2}$?" In 1 there are 6 sixths. In $\frac{1}{2}$ there are how many sixths? The answer is $\frac{1}{2}$ times 6, or 3. The problem is worked like this:

$$\frac{1}{2} \div \frac{1}{4} = \quad \quad \quad \frac{1}{2} \times \frac{4}{1} = 4$$

VI. DECIMAL FRACTIONS

1. Introducing Decimals

In order that the learning of decimals may proceed easily and expeditiously, the pupil must approach the study of decimals with an understanding of the fraction, with a knowledge of the significance of size and number of parts, and with the idea of ten and the use of position clearly in mind. To be launched upon his work with decimals, the pupil will need then only to review the things he has already learned and to be introduced to the conventional way of setting off the decimal from the whole number. Our discussion of decimals and our illustrations of typical sample

abstract, multiply, and divide units, you learned that tens, hundreds, and thousands were added, subtracted, multiplied, and divided just like units. So in your work now with decimals, you will find that tenths, hundredths, and thousandths are added, subtracted, multiplied, and divided just like units. In your earlier work in arithmetic, you found that you did not need to be thinking all the time you were doing something with tens, hundreds, and thousands about doing it just like units, but that if you were careful to put each part of the answer in its proper place that position would take care of everything for you. So in doing various things with tenths, hundredths, and thousandths, you will not need to be thinking about 'just like units,' but will only need to write each part of the answer in its proper place. In working with decimals, proper position will take care of everything for you.

Really, there is not anything new to learn in working with decimals, unless it is to be very careful about writing everything in its proper place, and that is not so new if you have already learned to be careful about numbers in their proper places.

Teaching the Writing and Reading of Decimals

11 is not 2 ones, but 1 ten and 1 unit.

111 is not 3 ones, but 1 hundred, 1 ten, and 1 unit.

1111 is not 4 ones, but 1 thousand, 1 hundred, 1 ten, and 1 unit -- and so on.

In the number, 1111, the second 1 to the left is ten times the first 1. It is in ten's position, and shows 1 ten. The third 1 is ten times the second 1. It shows ten tens, or 1 hundred, and is in hundred's position. The fourth 1 is ten times the third 1. It shows ten hundreds, or 1 thousand, and is in thousand's position. Each position to the left means ten times the position to the right of it.

Let us now start with the 1 thousand in 1111, and go to the right, seeing how the positions compare. The second 1 to the right of 1 thousand is 1 hundred, which is 1 tenth of the 1 to the left of it. The third 1 is 1 ten, which is 1 tenth of the 1 to the left of it. The fourth 1 is 1 unit, which is 1 tenth of the 1 to the left of it. Each position to the right means 1 tenth of the position to the left of it.

Let us now start with 1 unit and write 1 and 1 and 1 to the

right of it to show 1 tenth, 1 hundredth and 1 thousandth. We, however, write the 1's like this

1111

We can see at once that this is not the way to write 1 unit, 1 tenth, 1 hundredth, and 1 thousandth, because if so that means we 1 thousand, 1 hundred above. The usual method is to show that the first 1 on the left is 1 unit and that the next 1 to the right is 1 tenth. This is the way to write what we want.

1111

Notice the decimal point $.$ to the right of the 1 unit and to the left of the 1 tenth. The point $.$ is not a fraction. It is only a means of showing the position that is to be taken by the numbers that show tenths.

When we have the number

1111

we have 1 at the left of the point to show 1 unit. The next 1 to the right, next to the point and at the right of the point is 1 tenth of 1 unit, or 1 tenth, and so on tenth + position. The next 1 to the right is 1 tenth of the 1 on the left, that is, 1 tenth of 1 tenth, or 1 hundredth, and so on hundredth + position. The last 1 to the right is 1 tenth of the 1 on the left, that is, 1 tenth of 1 hundredth, or 1 thousandth, and so on thousandth + position.

Showing the Number and Size of Parts

In writing a fraction, we give full expression, we have to show the number of parts, and the size of the parts. In the writing of ordinary, or common, fractions, like $\frac{1}{2}$, for example, we write 1 as a numerator to show number of parts, and 2 as a denominator to show size of parts.

In writing decimal fractions, we also have to show the number of parts, and the size of the parts. In the writing of a decimal however, we write only one number, and that shows number of parts. But we have to have some way of showing size of parts also. In writing a decimal, we put the point where we

If, in writing a decimal, the number which we place to the right

of the point () that as we reach a position the position of the number that is written shows the size of the parts to be written

1 is read one tenth

2 is read two tenths

7 is read seven tenths

0.1 is read one tenth one tenth

25.2 is read twenty-five and two tenths

302.4 is read four hundred and two and four tenths

If, in writing a decimal, the number needs two places to the right of the point (), that is, in hundredths + position, the positions of the numbers written show the size of the parts to be hundredths.

15 is read fifteen hundredths

33 is read thirty-three hundredths

50 is read fifty hundredths

04 is read four hundredths (Notice that we use a zero to hold tenth's place when we have no tenths to write)

0.03 is read one and three hundredths

20.45 is read twenty and forty-five hundredths

If in writing a decimal the number needs three places to the right of the point (), that is, in thousandths + position, the positions of the numbers written show the size of the parts to be thousandths.

275 is read two hundred seventy-five thousandths

505 is read five hundred five thousandths

042 is read forty-two thousandths

0.007 is read seven thousandths (Notice that the zero is used to hold a place when we have nothing to write in it.)

500 is read five hundred thousandths

30.025 is read thirty and twenty-five thousandths

Notice that we read *and* only between the whole number and the decimal

Zeros written after a decimal do not change its value. $.5 = .50 = .500 = .5000$, and so on. Six dollars may be written as \$6 or as \$6.00.

To add a point () at the end of a whole number, followed by zeros, does not change its value. 3 may be written as 3.0, 3.00, 3.000. Usually we add no point (.), followed by zeros, at the end of whole numbers. Sometimes, however, it is convenient to do so when we are dealing with whole numbers and decimals, as we shall presently see.

$$\begin{array}{r} 230 \\ 500 \\ 375 \\ 975 \\ \hline 1,200 \end{array}$$
 Read the answer, 1 200. It is equally better to strike off the zeros that come at the end, 1 200 and give the answer as 12. Notice that we add and carry just as we do in adding whole numbers. Notice in adding the tenths column we get 12, which we know to be 12 tenths. Since we can write only one figure in tenths column, we write the 2 there and carry the 10, which is 10 tenths, or 1, to the unit's position, and write it there.

2. Let us subtract 925 from 75

$$\begin{array}{r} 75 \\ .925 \\ \hline \end{array}$$
 We may set the example down like this, but to aid the eye we attach a zero to the right of the 75, then our example will look like this:

$$\begin{array}{r} 750 \\ .925 \\ \hline 125 \end{array}$$
 Read the answer. Notice that we subtract and carry just as we do in subtracting whole numbers.

3. Let us subtract 675 from 75

$$\begin{array}{r} 750 \\ .675 \\ \hline .075 \end{array}$$
 To aid the eye we write the example as shown. We subtract and carry just the way we do in subtracting whole numbers. When we get to the tenths column, we subtract, "Six from six." It is necessary that we write the 0 in the tenths position in the answer, in order to keep the 7 and the 5 in their proper place. The 7 is hundredths and the 5 is thousandths; so we need a 0 in tenths place to make the 7 and the 5 show what they really are.

4. Let us add 20.5, 6.75, 14.825, and 167

$$\begin{array}{r} 20.5 \\ 6.75 \\ 14.825 \\ 16.7 \\ \hline 20.500 \\ 6.750 \\ 14.825 \\ 16.700 \\ \hline 58.775 \end{array}$$
 We can add as the example stands, but to aid the eye we attach zeros to fill the columns on the right before adding. Notice that we add and carry just as we do in adding whole numbers.

5. In the case of the...

17 1900 In the case of the...
18 1901 In the case of the...
19 1902 In the case of the...

6. In the case of the...

19 1900 In the case of the...
20 1901 In the case of the...
21 1902 In the case of the...
22 1903 In the case of the...
23 1904 In the case of the...
24 1905 In the case of the...
25 1906 In the case of the...
26 1907 In the case of the...
27 1908 In the case of the...
28 1909 In the case of the...
29 1910 In the case of the...

THE JOURNAL OF THE

The first part of the...
The second part of the...
The third part of the...
The fourth part of the...
The fifth part of the...
The sixth part of the...
The seventh part of the...
The eighth part of the...
The ninth part of the...
The tenth part of the...

THE JOURNAL OF THE

In the case of the...
In the case of the...
In the case of the...
In the case of the...
In the case of the...
In the case of the...
In the case of the...
In the case of the...
In the case of the...
In the case of the...

1. Method of...

20 1900 In the case of the...
21 1901 In the case of the...
22 1902 In the case of the...
23 1903 In the case of the...
24 1904 In the case of the...
25 1905 In the case of the...
26 1906 In the case of the...
27 1907 In the case of the...
28 1908 In the case of the...
29 1909 In the case of the...
30 1910 In the case of the...

2. Multiply 423 by 75

$$\begin{array}{r}
 423 \\
 75 \\
 \hline
 2115 \\
 2955 \\
 \hline
 31725
 \end{array}$$

Notice that we multiply and carry just as we do in multiplying whole numbers. When we multiply a number like 423 by a number less than 1 like 75, will the answer be more or less than the multiplicand?

3. Multiply 42.25 by 25

$$\begin{array}{r}
 42.25 \\
 25 \\
 \hline
 211.25 \\
 845.00 \\
 \hline
 1056.25
 \end{array}$$

Notice that we multiply and carry just as we do in multiplying whole numbers. We may strike off the last zero and give our answer as 1056.5.

4. Multiply 207 by 2.5

$$\begin{array}{r}
 207 \\
 2.5 \\
 \hline
 1035 \\
 734 \\
 \hline
 917.5
 \end{array}$$

Notice that we multiply and carry just as we do in multiplying whole numbers.

5. Multiply 25.75 by 8

$$\begin{array}{r}
 25.75 \\
 8 \\
 \hline
 206.00
 \end{array}$$

Notice that we multiply and carry just as we do in multiplying whole numbers. We may strike off the zeros in our answer, and give it as 206.

Notice these points:

(1) That in Example 1 we multiply *thousandths* by a whole number and get *thousandths*;

(2) That in Example 2 we multiply a whole number by *hundredths* and get *hundredths*;

(3) That in Example 3 we multiply *hundredths* by a whole number and get *hundredths*;

(4) That in Example 4 we multiply a whole number by *tenths* and get *tenths*;

(5) That in Example 5 we multiply *hundredths* by a whole number and get *hundredths*;

(6) That we use decimals in multiplication just as we use whole numbers.

Teaching the Multiplication of Decimals by Decimals

In multiplying decimals by whole numbers or by decimals, we can let position take care of our answers for us. Ten times tenths, or hundredths, or thousandths, multiplied by a whole number will give tenths, or hundredths or thousandths in the answer and a whole number multiplied by tenths, or hundredths or thousandths will give tenths, or hundredths, or thousandths in the answer.

But in multiplying a decimal by a decimal, although we multiply them just as though they were whole numbers, we cannot rely upon position to take care of our answers for us. We have to think for ourselves what the answer should be. Fortunately there is an easy rule to follow that will be our thinking for us. Let us see if we can find the rule.

1. Multiply 5 by 5

We multiply 5 by 5 as though we were multiplying whole numbers. We remember that we are multiplying tenths by tenths and that tenths multiplied by tenths give hundredths. So we have to put our answer down to make it show 25 hundredths.

$$\begin{array}{r} .5 \\ .5 \\ \hline .25 \end{array}$$

2. Multiply 25 by 25

We first multiply as though we were multiplying whole numbers. We notice that we are multiplying tenths by tenths, and that the answer must show hundredths. So we place the point of 1 in the answer for which it is 25. We notice that we multiply 2 and a little more by 2 and a little more. Since two times are four and answer is still a little more than 4. Neither 625 nor 625 is two times 25 or slightly more than 4. Only 625 is slightly more than 4.

$$\begin{array}{r} 25 \\ 25 \\ \hline 125 \\ 50 \\ \hline 625 \end{array}$$

3. Multiply 375 by 53

We first multiply as though we were multiplying whole numbers. We notice that our answer has to be in the neighborhood of 15, since we are multiplying 3 and a little more by 5 and a little more. It is not as high as 125 we are multiplying hundredths by tenths and we get thousandths. So we make our answer show thousandths. 19.875 must be the correct answer, since it is in the neighborhood of 15, and since it shows thousandths.

$$\begin{array}{r} 375 \\ 53 \\ \hline 1125 \\ 1875 \\ \hline 19875 \end{array}$$

We can by reasoning as thinking always decide just where to place the point in our answer. In Examples 1, 2, and 3 we have examined and paid where to place the point in the answer in each case. We know that our answers are correct. Now let us look at these examples and see whether we can find the rule.

In Example 1, count the places to the right of the decimal point in both multiplier and multiplicand together. There are two places to the right in both. Count the places to the right of the point in the answer. There are two places to the right of the point in the product. How do they compare?

In the same in Example 2. You find two places to the right of the point in the product, and two places to the right of the point in the multiplier and the multiplicand taken together.

In the same in Example 3. How many places to the right of the point in the multiplier and the multiplicand? How many places to the right of the point in the product?

Rule: In multiplying decimals, first multiply as you do with whole numbers. Next, point off in the product as many places as there are decimal places in both multiplier and multiplicand taken together.

4. The Division of Decimals

If the pupil has learned to divide tens, hundreds, etc., 'just like units,' by relying upon the position in which the various partial answers are written to aid in thinking, he will have no difficulty in learning to divide decimals. He proceeds, first, to divide decimals by whole numbers, and, next, to divide decimals by decimals. In the latter divisions, he may profit by the former if in each case he changes the divisor to a whole number. Such change in the divisor means that it has been multiplied by ten or by a power of ten, which requires that the dividend must be multiplied by the same number. Since the multiplications require nothing more than 'moving the point to the right' the required number of places, he will encounter no difficulty on that score. If, as he proceeds, he is conscious of what is required and of the purpose of each step of procedure, he will quickly learn to bring to bear upon this 'new' task the ideas and procedures he learned in previous lessons. The following para-

graphs will illustrate some of the various methods of procedure that may be followed.

Teaching the Division of a Decimal by a Whole Number

In dividing a decimal by a whole number, all we need to do is to place each part of the answer in its proper place. This is done exactly as we would when dividing a whole number by a whole number. If we are careful about position, position will take care of the answer for him.

1. Divide 52.75 by 25
2. Divide 5.275 by 25
3. Divide .5275 by 25

Notice how these three examples are worked

1.	2.	3.
$\begin{array}{r} 211 \\ 25 \overline{) 52.75} \\ \underline{50} \\ 27 \\ \underline{25} \\ 25 \\ \underline{25} \\ 0 \end{array}$	$\begin{array}{r} 211 \\ 25 \overline{) 5.275} \\ \underline{50} \\ 27 \\ \underline{25} \\ 25 \\ \underline{25} \\ 0 \end{array}$	$\begin{array}{r} .0211 \\ 25 \overline{) .5275} \\ \underline{50} \\ 27 \\ \underline{25} \\ 25 \\ \underline{25} \\ 0 \end{array}$

In each example, notice the position of each part of the answer. Notice that the point in each answer is just above the point in the dividend. Notice in the answer in Example 3 that we have to write a zero in tenth's place to hold that place. Is Example 1 how many 25's in 52.75? Is the answer 211 a reasonable one? Is the answer 211 or 21.1 a reasonable one? Is Example 2 how many 25's in 5.2 (that is, 52 hundredths)? Is 2 reasonable for the first quotient answer? In Example 3, how many 25's in 52 hundredths? Is 2 hundredths (that is, .02) reasonable for the first quotient answer?

Teaching Division by Decimals

In dividing by a decimal, since you already know how to divide by a whole number, it is a good plan right at the start to make the decimal in the divisor a whole number. Then you can do the moving the point the required number of places to the right. When the point is moved to the right, it is the same as multiplying by 10, or 100, or 1000, etc., as the case may be. So if you are dividing the divisor, you will need to balance things up by multiplying the

dividend by the same number. This you can do by moving the point an equal number of places to the right. Now, when you have made the dividend a whole number and moved the point in the dividend accordingly, you can divide as usual.

1. Divide 4.075 by 25.

2. Divide 4.075 by .25.

3. Divide 4.075 by .025.

Things to do:

(1) Make the dividend a whole number by moving the point to the right. Make a caret (^) to show where the point has been moved.

(2) Move the point in the dividend the same number of places to the right. Show by a caret (^) where the point has been moved.

(3) Divide as usual.

(4) Place the point in the quotient just above the caret (^) in the dividend.

1.	2.	3.
$ \begin{array}{r} 1\ 63 \\ 25 \overline{) 40.75} \\ \underline{25} \\ 157 \\ \underline{150} \\ 75 \\ \underline{75} \\ 0 \end{array} $	$ \begin{array}{r} 16\ 3 \\ .25 \overline{) 4.075} \\ \underline{25} \\ 157 \\ \underline{150} \\ 75 \\ \underline{75} \\ 0 \end{array} $	$ \begin{array}{r} 163. \\ .025 \overline{) 4.075} \\ \underline{25} \\ 157 \\ \underline{150} \\ 75 \\ \underline{75} \\ 0 \end{array} $

Teaching the Adding of Zeros in the Dividend

Adding zeros to the right of the decimal point does not change the value of a number. 25.8 is not changed in value when we make it 25.80, or 25.800, or 25.8000, etc. 885. is not changed in value when we make it 885.0, or 885.00, or 885.000, etc. Sometimes we find it convenient to add one or more zeros to the right of the point in the dividend in order to complete our division.

1. Divide .885 by .375.

We first make .375 a whole number by moving the point to the right, and next we move the point to the right in the dividend an equal number of places. We now divide. Our first quotient answer is 2.

$$\begin{array}{r} 2 \text{ } 36 \\ 375 \overline{) 883.00} \\ \underline{750} \\ 133 \text{ } 0 \\ \underline{112 \text{ } 5} \\ 22 \text{ } 50 \\ \underline{21 \text{ } 50} \end{array}$$

When we multiply and subtract, we have a remainder of 135. That means nothing more to the right of the point than a 0. If we get the number of numbers, we write no more digits to the right of 585, as we found in multiplication says that we place the point in the quotient just after the first 5 in the dividend, and our answer is 2.36.

2. Divide 171 by 54

$$\begin{array}{r} 3 \text{ } 100 \\ 54 \overline{) 171.000} \\ \underline{162} \\ 9 \text{ } 0 \\ \underline{54} \\ 3 \text{ } 60 \\ \underline{324} \\ 360 \\ \underline{324} \end{array}$$

We first make 54 a whole number by moving the point in the right, and then we move the point in the dividend an equal number of places. We now divide 171000 by 54. When we multiply and subtract, we have a remainder of 9. The nothing more to the right of the point in the dividend, we can continue to divide. In this particular example we could keep on adding zeros in the dividend and dividing, and still have a remainder. If we wish to stop, answer for hundreds hundredths, only we must divide as we have done until we get our quotient answer for hundredth's place, which we write as 3.17. If we wished to be still more exact by including thousandths we would add another zero in the dividend and find the quotient figure for thousandth's place. Our answer would be 3.167, which we would write as 3.167.

3. Divide 1185 by 625

$$\begin{array}{r} 1896 \\ 625 \overline{) 1185.000} \\ \underline{625} \\ 5600 \\ \underline{5000} \\ 6000 \\ \underline{5625} \\ 3750 \\ \underline{3750} \end{array}$$

We change the divisor to a whole number by moving the point in the right, and then we move the point in the dividend an equal number of places. In order to multiply 625 by 18 we must move to the right of the point in the dividend 18 times. We wanted our answer in a dividend of four places hundredths, we would have to keep dividing when we had 1896 as the answer and notice that 1896 is 1896000 a power of ten more than 5, we would have to move our answer point 30.

It is seldom necessary to carry the answer far

1000 hundredths' place, since thousandths are very small parts. When we reach our answer in the sixth record, we carry our work to the thousandths' place, and if the number in that place is less than 5, we strike it off. If it is 5 or more, we drop it and add 1 to the figure we have in thousandths' place.

The point to remember is that we can add as many zeros as we need to the right of the point in the dividend.

2. Divide 54.6 by 136

$$\begin{array}{r} 54.6 \\ 136 \overline{) 54.600} \\ \underline{408} \\ 780 \\ \underline{704} \\ 760 \\ \underline{704} \\ 560 \\ \underline{544} \\ 160 \\ \underline{136} \\ 240 \\ \underline{272} \\ 680 \\ \underline{680} \\ 0 \end{array}$$

We first change the divisor to a whole number by moving the point to the right. We next move the point in the dividend to the right an equal number of places. In order to do that, we have to add two extra zeros to provide enough places. We now divide as usual, and place the point in the quotient just above the point in the dividend.

Teaching Fractional Equivalents

Equivalent is a double word. The part *equi* means equal; the part *valent* means value. *Equivalent* means "of equal value." Let us find the equivalents of some of the common fractions you have learned to use.

One of the things that you learned about a fraction when you began to study them was that a fraction means to divide. $\frac{1}{2}$ shows that 1 has been divided into 2 equal parts. $\frac{1}{4}$ shows that 1 has been divided into 4 equal parts, and so on. When such fractions as $\frac{1}{2}$, $\frac{3}{8}$, $\frac{1}{4}$, etc., are shown, we can find the answer by dividing, thus:

$$\begin{array}{r} 3 \\ 4 \overline{) 12} \\ \underline{12} \\ 0 \end{array}$$

$$\begin{array}{r} 3 \\ 8 \overline{) 24} \\ \underline{24} \\ 0 \end{array}$$

$$\begin{array}{r} 6 \\ 7 \overline{) 42} \\ \underline{42} \\ 0 \end{array}$$

Now that you have learned how to divide a smaller number, like 3, for example, by a larger number, like 4, for example, by adding zeros to the right of the point in the dividend as they are needed, you are able to find the equivalents of common fractions, like $\frac{1}{2}$, $\frac{1}{4}$, etc., by dividing.

We know that 1 can be written as 1.0, or 1.00, or 1.000; that 3 can be written as 3.0, 3.00, or 3.000, etc. Let us then in dividing

add a point (.) followed by enough zeros to make up the fraction for the denominator

$$\begin{array}{r} 3 \\ 2 \overline{)10} \\ 6 \\ \hline 40 \\ 20 \\ \hline 20 \\ 20 \\ \hline 0 \end{array} \qquad \begin{array}{r} 30 \\ 2 \overline{)60} \\ 40 \\ \hline 20 \\ 20 \\ \hline 0 \end{array} \qquad \begin{array}{r} 300 \\ 2 \overline{)600} \\ 400 \\ \hline 200 \\ 200 \\ \hline 0 \end{array}$$

Thus we divide and find that $\frac{3}{2}$ is the same as 1.5, or 1.50 or 1.500

Usually we just make the one denominator, 200, for instance, and $\frac{3}{2}$ equals 1.50. In using 5 with other denominators, we write the whole as hundredths, or thousandths, we can easily change it by adding zeros: .50, or .500

$$\begin{array}{r} 25 \\ 4 \overline{)100} \\ 8 \\ \hline 20 \\ 20 \\ \hline 0 \end{array} \qquad \begin{array}{r} 300 \\ 3 \overline{)900} \\ 600 \\ \hline 300 \\ 300 \\ \hline 0 \end{array} \qquad \begin{array}{l} 57 \text{ or } 5.7 \text{ or } 5.70 \text{ } \frac{57}{10} \text{ is about} \\ \text{the same as } \frac{3}{2} \text{ } \text{To be sure, we} \\ \text{may write it } \frac{3}{2} = 1.5 \end{array}$$

Since $\frac{1}{2}$ is the same as .5 or .50, $\frac{1}{2}$ of a number is the same as multiplying it by .5 or .50.

$$\begin{array}{r} 48 \\ .5 \\ \hline 24.0 \end{array} \qquad \frac{1}{2} \text{ of } 48 = 24 \qquad \begin{array}{r} 24 \\ 240 \\ 4 \\ 8 \end{array}$$

Since $\frac{1}{3}$ is the same as .33 $\frac{1}{3}$, $\frac{1}{3}$ of a number is the same as multiplying it by .33 $\frac{1}{3}$.

$$\begin{array}{r} 48 \\ .33\frac{1}{3} \\ \hline 16 \\ 144 \\ 144 \\ \hline 16.00 \end{array} \qquad \frac{1}{3} \text{ of } 48 = 16 \qquad \begin{array}{r} 16 \\ 3 \cdot 48 \\ 3 \\ 14 \\ 16 \end{array}$$

VII PERCENTAGE

I The Hundredth Part

Immediately following the study of decimals, percentage should be introduced as another way of writing and speaking of the hundredth. Instead of being widely separated from the study of decimals, percentage should be intimately related to decimals. The pupils should be made to feel that they are undertaking nothing new other than the learning of the method of writing and speaking of the decimal fraction -- hundredth -- commonly used in business and practical affairs.

Percentage is a continuation of fractions. It is a special case of the subject and it may be made to afford excellent practice in enlarging the ideas of fractions and in securing greater facility in using them. The following illustration will serve to show the language change in the transition from fractions to percentage. A man had 700 chickens and sold 1 out of every 2, or 1 out of every 10, or 1 out of every 50, or 1 out of every 100, or 1 out of every 350, how many were sold? The above problem would be classified under the subject of fractions. Since 1 out of 2 was sold, the number sold was $\frac{1}{2}$ of 700, or since 1 out of 10 was sold, the number sold was $\frac{1}{10}$ of 700, or since 1 out of 100 was sold, the number sold was $\frac{1}{100}$ of 700.

If we agree that the word *per* shall mean *out of* when used in this connection, the above statements would be: 1 per 2, or 1 per 10, or 1 per 100. If we substitute the Latin word *decem* for *ten*, our statement becomes 1 per *decem*; if we substitute the Latin word *centum* for *hundred*, our statement becomes, 1 per *centum*. If we abbreviate the word *centum* by cutting off the last two letters we have 1 per cent. We frequently abbreviate words in this way in arithmetic; for example, we write *int.* for *interest*, and *frac.* for *fraction*. Therefore 1 per cent means 1 out of every 100. If a problem involving 1 out of every 100 is properly classified as a problem in fractions, it is certain that a problem involving 1 per cent

is also a problem in fractions. Percentage enters in as a separate topic almost the beginning of the fractional structure.

The fact that a quantity is measured off parts, hundredths instead of into any other possible number of parts, requires to be no valid reason for introducing percentages as a new phase in the development of numbers.

The following sample materials are introduced to exhibit and illustrate the introduction of percentages as a special phase of the general topic of fractions.

Teaching the Relation of Percents to Fractions

In your work with fractions you studied almost parts of various different sizes, such as tenths, thirds, fourths, fifths, hundredths, etc. In your work with decimals, you studied almost parts of only three or four different sizes, such as tenths, hundredths, and thousandths, and occasionally almost parts as small as ten thousandths. In the work you will now begin, you will learn to study almost parts that are of one special size—hundredths.

In your work with hundredths, you will learn really nothing new to learn about parts, since you already have learned or know how to work with decimals how to add, subtract, multiply, and divide hundredths. The only reason why you will now have to make a special study of hundredths is that hundredths have a special use in, and are now and everyday life. In business and in everyday life hundredths are spoken of, written, referred to, and used very extensively, as are the other decimals, like tenths and thousandths, and you must know how they relate to, or as they connect with, hundredths.

In business and in everyday life hundredths are spoken of as cents, and referred to as percent. (Percent is a strange term which means hundredth. One does not think of the word until it is used. One uses the term 'percent.' One does not think of a dollar until it is used. One uses the term 'dollar.' A cent is a coin. It is called a cent because it is a hundredth part of a dollar. Percent means hundredth when applied to anything. A 'cent' refers to money only. Percent refers to the size of a part of anything that is a hundredth. Do not confuse percent with dollars and cents.

¹ J. C. Brown and L. D. Coffman. *The Teaching of Mathematics*. Houghton, Mifflin and Company, Chicago, 1924, pp. 214-215.

Instead of writing the word 'percent,' the sign % is usually written. Thus 75% is read "75 percent."

Illustrating Common Uses of Percent

1. The children had 100 words to spell in a review lesson. Robert spelled 87 of them. Since he spelled 87 of all the words, he was given a grade of 87%.

2. In his history work last month, Sam did only 80, or 80, as well as he should have done. The teacher gave him a grade of 80% in history.

3. Last month nine-tenths of all the children in the fifth grade were neither absent nor tardy. This was, of course, 90 of them. And so, when the report went in, the fifth grade was mentioned as having 90% of the children neither absent nor tardy.

4. Mr. Simpson bought some cement to use in putting a floor in his garage. When the dealer sent the bill for it, he said he would take 10% off for cash. Mr. Simpson paid cash for the cement, and so his bill was one-tenth, or 10%, less than if he had waited until the next month to pay the bill.

5. The Wright Clothing Store had the following advertisement in the papers: "Summer Suits, 25% off." What was meant was that the price of summer suits had been reduced one-fourth, or .25, or 25%.

6. The College Book Store received a bill from the publishing company for books bought. The bill showed that the Book Store owed \$30.78. The bill was marked "Discount, 20%." This meant that the price charged, \$30.78, had been reduced one-fifth, or .20, or 20%.

7. John's brother had to borrow \$150 in order to help pay his expenses at college. He agreed to pay 6% of the \$150 each year for the use of the money. That is, he agreed to pay .06 of the \$150 each year for the use of the money.

8. The Creamery Company tested the milk it was buying, and found that .04 of the milk was butter fat. In keeping a record of the test, the clerk wrote it like this: Butter fat = 4%.

9. 60% of the crowd at a football game were high-school students. This means that .60, or 60, of the crowd were high-school students.

10. Last fall the local high-school football team won 50% of all

games played, lost 30%, and lost 20%, "What is percent for the expression 30%? How much is 30%? It is 30% of 30%."

11. James was applying to his own ideas of percent. He said, "I try to make 45%, and what I am making. He means that he tried to make 45% of what he made."

12. Mr. Johnson was making 20%, and then making to make 20% a home. That is, he was making 20% of 20%, and then making 20% of 20%.

Changing Percents and Denominals

When one has a percentage in percent rather than in decimal life, he finds that the fractional parts that he has do not seem related not as a decimal, but as percent. In making these percentages, one can have learned from the same denoms. he does not know that percent that is given in a decimal. The question is, how much is 30% of 30%? The answer is a percentage in percent. In making the percentages, one gets the answer first as a decimal. He does not know that percent and to percent. But we now have the change over to the decimal.

(1) The way to change a percent to a decimal is to drop the percent sign, %, and move the decimal point two places to the left. A rule is given when necessary.

$$25\% = .25$$

$$60\% = .60$$

$$11\% = .11$$

$$75\% = .75$$

(2) The way to change a decimal to a percent is to move the decimal point two places to the right, and add the percent sign, %. A rule is given when necessary.

$$.14 = 14\%$$

$$.2 = 20\%$$

$$.03 = 3\%$$

$$.015 = 1.5\%$$

A Table of Equivalents

Sometimes it is helpful in thinking "just how much is 30% of anything?" to think that 30% is the same as $\frac{3}{10}$ or $\frac{30}{100}$, and how much is 25% of anything? to think that 25% is the same as $\frac{1}{4}$, and so on.

Here is an important table of equivalents. It will be helpful as you learn all of them.

THEORY OF PAIRS

100%	= 1	12 1/2%	= 1	40%	= 1	8 1/2%	= 1/2
32 1/2%	= 1	27 1/2%	= 1	60%	= 1	6 1/2%	= 1/2
60 1/2%	= 1	62 1/2%	= 1	70%	= 1	5 1/2%	= 1/2
20%	= 1	67 1/2%	= 1	10 1/2%	= 1	4%	= 1/2
75%	= 1	20%	= 1	64 1/2%	= 1	2 1/2%	= 1/2

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[illegible][illegible]

2. In accordance with the report made by the Director of the Bureau of the Census, the Bureau of the Census has been authorized to conduct a study of the economic conditions of the United States and to report thereon to the President and the Congress.

[illegible]

The systematic study of general language really is a combination of the part in its relations to the whole, and especially to such as the ability to deal with a text from a point of view of the corresponding ideas and their general and specific character and the characters of the text. The purpose of this study is to find out the general principles of the language and the ability to deal with a text from a point of view of the corresponding ideas and their general and specific character and the characters of the text. The purpose of this study is to find out the general principles of the language and the ability to deal with a text from a point of view of the corresponding ideas and their general and specific character and the characters of the text.

the pupil acquires familiarity with the fraction and with the rules that govern its use. In dealing with it as an independent abstraction, he acquires facility in dealing with it as it may be applied to anything and everything.

The relational idea of the fraction is, however, never completely lost. The necessity of giving attention to the size of the part enforces attention upon relations. Moreover, the practical applications that the pupil is called upon to make from time to time call attention to relations. In the midst of, or immediately following, his study of each of the three ways of dealing with parts—namely, common fractions, decimals, and percentage—the pupil is introduced to the applications that are commonly called "the three kinds of problems." These are:

- (1) Finding the part, or percent, of a number;
- (2) Finding what part, or percent, one number is of another;
- (3) Finding a number when a part, or percent, of it is given.

The three kinds of problems are called 'problems in fractions' simply for the pupil's convenience. They are in reality methods of studying relations between numbers. They serve to carry the pupil back, after his excursions into the study of parts, to a more complete, as well as a more useful, study of groups than he has heretofore been able to pursue. After learning to deal with the part as an independent abstraction, the pupil receives training by means of the three kinds of problems in dealing with groups upon a higher level than has hitherto been possible without the use of the fractional idea.

II. A CONTINUATION OF EARLIER STUDIES

Studying the relations between numbers is no new activity for the pupil. In the first grade he engaged in the comparison of groups; later he analyzed certain larger groups into smaller equal groups and noted the number of smaller groups resulting from the analysis; still later he extended his studies

of equal groups in the finding of how many times one number is another number. After treating this case in the previous volume the process of division as a process of partitioning is treated recently. The pupil has been learning already how to divide in fractions, decimals and percentages. Some of the machinery of dealing more rapidly with the relations between numbers. The volume also gives an introduction to the three kinds of proportions. It is not a completely thorough work on entirely new material. It is merely brief and understandable enough to explain and more exact display of material that has been brought to his concern from the student's own studies of arithmetic.

As indicated in Chapter VIII the three kinds of proportions are often introduced as three different methods of comparison, and each upon their separate consideration, and then they give the opportunity of their separate treatment. It is for the expected that the so-called 'proportion' will appear as four or ways of finding answers, but if not, as combinations in the problems and contrast with the other two and will help the student to develop an understanding of relationships between numbers. Moreover, if the attempt is made to teach that the pupil already knows about the relations between numbers, the means of introducing the new ways of extending relations, the new ways will appear less and less as methods of comparison and more and more as different ways of dealing with number relations and speaking about them.

III The Kanish Proportion

The earlier presentation of the three kinds of proportions follows the study of common fractions and as such for the purpose of introducing the various methods of comparison required in the three solutions. The later presentation made in connection with decimals and percentages are designed to give later opportunity for the use of the methods of comparison, are especially useful in making these more understandable through comparison and contrast. As though comparison and contrast between the three proportions

are important from the beginning, and a great deal can be undertaken until the pupils develop an understanding of the method of performance that belongs with each kind of problem. Thus, at the outset each problem must be considered pretty much as a special case. Later each problem is considered in its relation to the others.

In connection with their work in the multiplication of fractions, the pupils have brushed almost the 'problems of the first kind,' for example.

Find $\frac{1}{2}$ of 25

A yard of dress goods costs \$1.60 How much will $\frac{1}{2}$ of a yard cost?

All the pupils need is to classify the kind of multiplication indicated under its appropriate designation. They can quickly proceed to 'problems of the second kind,' which may be presented somewhat as follows:

*Presenting Problems of the Second Kind Finding What Part
One Number Is of Another Number*

The purpose of this kind of a problem is to compare one number with another, or to see how one number is related to another. In this kind of problem, one number is first said what part a smaller number is of a larger one.

Let us compare 5 with 15. That is, see how 5 is related to 15, or find what part 5 is of 15.

A smaller number, like 5, must always be a fractional part of a larger number, like 15. It cannot be 3 times 15, or 2 times 15, or 1 times 15; it must be a part of 15, since it is smaller. Therefore, one should remember that when the question, "what part?" is asked, the answer must be expressed, not as a whole number, but as a fraction. Since part means fraction, the question, "what part?" suggests that the answer wanted is a fraction.

The question may be asked either way, as follows:

- | | |
|--------------------------|---|
| 1. 5 is what part of 15? | } What part suggests that the answer
is a fraction |
| 2. What part of 15 is 5? | |
| 3. What part is 5 of 15? | |

We already know that $\frac{1}{2}$ of 12 is 6 and that 6 is one half of 12. Let us now know the same thing for any number n which $\frac{1}{2}$ has above $\frac{1}{2}$. Then we also know

When we divide a number by 2, if we do not get an even number by dividing the number by 2, the answer is a fraction whose denominator, and numerator are the same and just halved the original number by a larger value, we obtain a fraction by $\frac{1}{2}$ and a denominator, then $\frac{1}{2}$.

The fraction $\frac{1}{2}$, when reduced, is $\frac{1}{2}$.

In the presentation of the 'Marshes of Flatland' and generalization of contrast which the number words 'three' and 'four' have the goal to distinguish the difference. We are going to use the problem:

"James owned 12 of his potatoes which were $\frac{1}{2}$ of all the food he had bought for his family. If he had bought 12 more potatoes, how many would he have bought?"

The goal is to find the number which is the answer to the problem. We will have the food $\frac{1}{2}$ of 12 and we will have the answer by dividing 12 by 2 (the number). When the number 12 is divided by 2, we get that a difference of 6 is made in the number of 12 and 6 is indicated, for every 12 we have 6 and we have 6 and 6 is the answer of procedure. The answer is indicated by the number 6.

Presentation of the Problem of James' Potatoes

James is a farmer who has 12 potatoes.

1. James had bought 12 potatoes which were $\frac{1}{2}$ of all the food he had bought for his family. If he had bought 12 more potatoes, how many would he have bought?

Notice what the problem tells. James' potatoes were $\frac{1}{2}$ of all the food he had bought for his family, for 12 potatoes.

Notice how it says for James

If 12 parts of all the potatoes — 12 potatoes.

Then 1 part of all the potatoes — $\frac{1}{2}$ of 12 is 6 potatoes.

and 6 parts of all the potatoes — 6 is 12 is all potatoes.

2. Edith had just finished reading 240 pages of Part Two. She said, "I have finished reading about $\frac{2}{3}$ of the book." If that is so, about how many pages are there in the entire book?

If 2 parts of the entire book = 240 pages,

then 1 part of the entire book = $\frac{1}{2}$ of 240, or 120 pages,

and 3 parts, or the entire book = 3×120 , or 360 pages.

3. Helen was looking at a new dress that cost \$12.00. She thought, "If I buy this, it will take $\frac{3}{4}$ of all the money I have." How much money had she?

If 3 parts of Helen's money = \$12.00,

then 1 part of Helen's money = $\frac{1}{3}$ of \$12.00, or \$4.00,

and 4 parts, or all Helen's money = $4 \times \$4.00$, or \$16.00.

4. $\frac{1}{3}$ of a certain number is 14. Find the number.

1 part of the number = 14,

3 parts of the number = $3 \times 14 = 42$.

5. $\frac{2}{3}$ of a certain number is 36. Find the number.

If 2 parts of the number = 36,

then 1 part of the number = $\frac{1}{2}$ of 36, or 18,

and 3 parts, or all the number = 3×18 , or 54.

Whenever you have given the fractional part of a number, you can always find the number by thinking of what you have just been shown and as you have been doing.

There is a shorter way of finding the number when a fractional part of it is given that does not require so much thinking or so much writing. It is the *method of division*.

Turn back to Problem 1, and read it. Notice how it may be solved:

$$\begin{array}{r} 6 \\ 12 \div \frac{2}{3} = 12 \times \frac{3}{2} = 36 \end{array}$$

Read Problem 2, and notice:

$$\begin{array}{r} 120 \\ 240 \div \frac{2}{3} = 240 \times \frac{3}{2} = 360 \end{array}$$

Read Problem 3, and notice:

$$\begin{array}{r} 6.00 \\ \$12.00 \div \frac{3}{4} = \$12.00 \times \frac{4}{3} = \$16.00 \end{array}$$

Notice Example 4

$$14 + \frac{1}{2} = 14 + \frac{1}{2} = 29$$

Notice Example 5

$$12 \\ 26 + \frac{1}{2} = 26 + \frac{1}{2} = 51$$

Following the presentation of the three kinds of problems, as indicated above, the pupil should be given simple problems with each kind separately and with all three kinds together. The outcome of the presentation and the problems should be (1) the ability to distinguish the difference between the first and third kinds of problems, (2) the understanding of the method of discussion used in solving the third kind and (3) the ability to indicate one number as the fractional part of another. The ability and the understanding indicated will not be fully developed, and the pupil will need help and guidance in making the necessary distinctions. It will receive, however, a type of presentation for a considerable number of similar studies and for the continued study of the other three kinds of problems in connection with fractions and percents.

15. Later Presentation

In the later presentation the pupil has the benefit of having learned the methods of performance suggested in problems of the first and third kinds, as well as of having learned the method of comparing the indicated amounts when one number is stated as the fractional part of another. The methods of performance used first can be suggested as those the pupil may proceed with them at his own pace. The comparisons and contrasts that are necessary for being able to an independent mastery of all three kinds of problems. The methods suggested in what follows may be used in conjunction with the three kinds of problems in fractions and percents. The contrast between the first and the third kinds will help to bring each to the level of understanding.

ing, and the corresponding whole numbers in comparison with the second kind will help the pupil to understand more completely the whole number when he finds in this what governs a larger number as well as a smaller one and afterwards a solution by dividing the smaller by the larger as is usually required. The pupil should be made aware from the outset of such a possibility that he is undertaking to study the same three kinds of problems he studied in the earlier part of the course.

Teaching the Comparing of 2 numbers

The methods of comparing any two numbers are straightforward and obvious. When one number is first said "how many times" as "how much less" one number as than another for some comparison. When one number is first said "what part of" or "how many times" one number as another, the same direction.

The second kind of fractions or decimals problems that you have just been treating were chosen to find what part one number is of another. In this kind of problem you were comparing a smaller number with a larger one to find what part the smaller is of the larger.

A very similar kind of comparison is made when one compares a larger number with a smaller one. The question is the answered as "how many times" the smaller is the larger. The method is to divide the larger by the smaller.

In these two kinds of comparison problems the method is obvious. In both kinds the number being compared is being asked about, is put in the numerator as he described. In both kinds the number which goes in the denominator when the question is asked, "what part of" or "how many times" is put in the denominator, to be used as the divisor.

When the question "what part of" is asked, the answer is a fraction, or a decimal. When the question "how many times" is asked, the answer is a whole number or a mixed number.

Let us compare a smaller number with a larger one.

1. 6 is what part of 157

$$\frac{6}{157} = \frac{1}{26} \quad \text{or} \quad \frac{6}{157} = 157 \overline{) 6.0} \\ \underline{60} \\ 00$$

3. What percent of 144 is 112? The question asks about 112; so we put 112 in the dividend

$$\begin{array}{r} 144 \overline{) 112.00} \\ 112 \\ \hline 0 \end{array}$$

4. What percent of 112 is 140? The question asks about 140; so we put 140 in the dividend

$$\begin{array}{r} 112 \overline{) 140.00} \\ 112 \\ \hline 28 \\ 22 \\ \hline 60 \\ 56 \\ \hline 40 \end{array}$$

In Example 1, we compare 6 with 24

In Example 2, we compare 24 with 6

In Example 3, we compare 112 with 144

In Example 4, we compare 140 with 112

In working any example or problem in which two numbers are to be compared by division, always ask yourself one of these questions: (1) "About which number does the question ask?" (2) "Which number is the one being compared with the other?"

Tracking the Solution of Mixed Problems

Let us compare the three kinds of problems in percent:

1. Bettie has 96 books all her own in her library at home. 75% of them are storybooks. How many storybooks does Bettie have? (First kind)

2. Bettie has 96 books all her own in her library at home. 72 of them are storybooks. What percent of all Bettie's books are storybooks? (Second kind)

3. Bettie has 72 storybooks in her library at home. This is 75% of all the books in her library. How many books does Bettie have in her library? (Third kind)

In Problems 1 and 3, a percent and a number are given, and you

are asked to find the other number. In Problem 2 two numbers are given, and you are asked to find the percent. It will be easy, therefore, to note the difference between a problem like Problem 1 and a problem like either of the others, and to recognize a problem like Problem 2. In Problem 2 you are asked to find *what percent* the 72 storybooks, 72 is the number being asked about. In any problem like Problem 2, you must put the number being asked about in the dividend. In any problem, when two numbers are given and you are asked to find the percent, always ask yourself first of all, "About which of the two numbers is the question being asked?" Then you will know which one goes in the dividend.

Problems 1 and 3 are easy to tell from Problem 2, but sometimes they are not so easy to tell from the others. In each a number and a percent are given, and in each you are asked to find the other number. You know, of course, that in Problem 1 the other number is found by multiplying by the percent, and in Problem 3 the other number is found by dividing by the percent. But we now have to tell Problem 1 from Problem 3.

Notice in Problem 1 that it says "75% of them" (meaning the 96 books) of a number that is given, and in Problem 3 that it says "75% of all the books," which is of a number that is not given, but is to be found. If you can see this difference between Problem 1 and Problem 3, you can make good rules for solving these problems and all others like them.

Rule When a percent is given of a number that is given, multiply by the percent. (First kind)

Rule When a percent is given of a number that is not given, divide by the percent. (Third kind)

Rule When two numbers are given to find the percent, put the number being asked about in the dividend. (Second kind)

Notice how we follow these rules in making Problems 1, 2 and 3.

Problem 1

96
75
—
4 80
67 2
72.00

Problem 2

75, or 75%
96 ÷ 75 (100)
67 2
4 80
4 80

Problem 3

100
75, or 75%
—
67 2
4 80
4 80

In the practice exercises that follow the illustrations, the 'problems' should be presented in mixed order, so that the pupil will be required, when he chooses his work, to distinguish its characteristics and to decide upon its kind. Very beneficial practice may be had in the requirement that, when the pupil has solved a problem of a given kind, he should use all the facts at hand and turn it into a problem of each of the other two kinds. The general directions given the pupil for attacking the practice exercises should be such as to aid him in making distinctions. They may take the form illustrated below. They should be supplemented by the teacher according to the pupils' needs.

Illustrative Problems

These problems with percent are not all alike. Some are problems of the first kind, some are problems of the second kind, and some are problems of the third kind. Read each problem carefully, so that you will know exactly what is to be done. If there is a percent and a number are given, and a question is asked for the number, look at the percent and the question and see if the percent is of the number that is given or if the number is of the percent, but to be found. When the numbers are given and a question is asked for the percent, look at the numbers and the question and see if the problem is making a part of making a whole or the whole of a part.

1. There are 24,000 books in the city library. Of these, 2000 are books on history and geography. What percent of the books in the library are books on history and geography?

2. In a certain school library there are 17,000 books. Of these, 65% are storybooks. How many storybooks are there in this school library?

3. Farmer Brown has found that about 80% of the apples he picks from the trees are good apples. He has just received an order from a city merchant for 40 bushels of good apples. How many bushels will he have to pick from the trees to get enough to fill the order?

Which is a problem of the first kind? Which is one of the second kind? Which is one of the third kind?

1a. 20 is how many times 2? Now you can make the second part of the problem read

1b. If a certain number of oranges cost 5¢, how much will 10 times that number cost?

Here are the two solutions of Problem 1

<i>Solution I</i>		<i>Solution II</i>	
Finding the cost of one		Finding the relation between numbers	
$2\frac{1}{2}$	$2\frac{1}{2}\text{¢}$	10	5¢
$2)\overline{5}$	20	$2)\overline{20}$	10
	40		50¢
	10		
	50¢		

In many problems Solution II is the easier. Often you can do most of Solution II 'in your head.'

2. If bananas are selling at 30¢ a dozen, how many can be bought for 5¢?

<i>Solution I</i>		<i>Solution II</i>
$3\frac{1}{2}$		
$12)\overline{30}$	$5 \div 2\frac{1}{2} =$	$\frac{5}{2\frac{1}{2}} = \frac{1}{2}$
24	$5 \div \frac{1}{2} =$	$\frac{1}{2} \text{ of } 12 = 6$
6	$5 \times \frac{2}{1} = 2$	
$12 = \frac{1}{2}$		

Often one does not need pencil and paper, if he can just see the relation between the numbers. In solving Problem 2, one could think the solution to himself, like this: 3¢ is $\frac{1}{2}$ of 6¢; $\frac{1}{2}$ of 12 is 6.

3. Last week Mrs. Jones bought $2\frac{1}{2}$ yards of muslin for 90¢. Now she thinks she can use 10 yards more. How much will the 10 yards cost?

Ask yourself, 10 is how many times $2\frac{1}{2}$? If you can see that 10 is 4 times $2\frac{1}{2}$, you can easily find the answer by multiplying 90¢ by 4.

4. If the price of lemons is 40¢ a dozen, how much will 3 lemons cost?

Ask yourself, 3 is what part of 12? If you can see that 3 is $\frac{1}{4}$ of a dozen, you can easily find the answer.

6. If twenty pencils will fill a box, how many will fill another box of 24 cents?

Ask yourself, 24 is how many times 12? If you can find the relation between 24 and 12 you can easily find the answer.

6 Mrs. Stewart remembered that when she bought five bags of corn she had the 12 pounds of corn she used every day. How much corn did she make a larger bag, and how many pounds of corn? How much will the 12 pounds of corn cost?

Ask yourself, 12 is how many times 12?

In the pupil's earlier work in studying problems, the work directed by the questions of the teacher and of the book are "What part of percent of 24 is 6?" "How many times 6 is 35?" etc. In the exercises just presented the pupil is required to answer the same kinds of questions. But instead of being asked by the teacher or book the questions are asked by the pupils and with which he has to deal. It can never practical or important the mathematics way for they serve the very practical purpose of reinforcing upon the pupil the necessity of looking for the questions which are his independent activity. These situations serve to transfer the practice upon the 'mental habit of problem' from the board

CHAPTER XIX

THE APPLICATIONS OF PERCENTAGE

ANALYSIS

1. There are many situations in which the idea of percent is an element that the pupil should study.

2. Such situations involve other elements that are unknown to the unsituated. To learn the situations, such other elements must become familiar.

3. Such other elements do not illustrate or apply percentage, an understanding of percentage helps to make the other elements understandable.

4. To view such situations merely as means of developing or applying computational ability is to miss their real worth.

5. Each situation should be studied as a matter of importance in its own right.

6. The idea of percent is helpful both in making a given situation understandable and in establishing relations between various situations that otherwise are not related.

7. Lesson materials are offered to illustrate:

(a) The method of studying the various elements of a given situation.

(b) The way the idea of percent, as represented in the 'three kinds of problems,' may be used to relate and classify otherwise diverse situations.

I. THE STUDY OF SITUATIONS

From the activities set up for the purpose of developing an understanding of percentage as the expression of relations between quantities, the pupil proceeds to the study of the personal, practical, and human situations of life in which the form of percentage is used as a means of expression and

raise. Having learned percentage, which is a characteristic of all these situations, or an idea of relations that makes them intelligible, or a method of statement that serves to describe them, the pupil is ready to proceed to a study of the other characteristics of the various applications that are intimately related to the idea of percentage and stated in the language of percentage. These 'other characteristics' are not at first mere illustrations of percentage, indeed, they cannot serve as illustrations of anything until they are known. They are, rather, activities of the larger life surrounding the pupil in which he has not as yet been engaging. His introduction to these other characteristics is facilitated by his knowledge and understanding of percentage, which is intimately a part of all of them.

II. COMPUTATIONAL ABILITY VERSUS UNDERSTANDING

The 'applications of percentage' are frequently presented as methods of finding answers to the 'problems' they serve to classify. Having learned to compute in terms of percents, the pupil is shown how answers are computed, first in this 'application,' then in that one. It seems as if those who follow this type of teaching believe that the pupil will be called upon to spend his days figuring interest on notes, determining the amount of discount on bills, computing the premiums on his fire and life insurance policies, finding the tax levy, and summing up the tax forms.

The truth of the matter is that one very rarely has to figure the interest on a note, or determine the amount of discount, or compute the premiums upon insurance policies, or decide exactly what the tax levy shall be. All such computations are made for one by the companies, firms, associations, and levying bodies with which one deals; and the computations are made in accordance with tables and charts prepared for such purposes. It is true, of course, that in one's personal and business budgeting he must be able to determine costs in advance or to check the correctness of

various kinds of bills presented to him for payment. Just such computations and such checking persons find difficulty so far as the computational and checking processes are concerned. All one needs to do in such matters is to recognize the four fundamental processes with brevity. The essence of the total procedure, however, is understanding, and without understanding one is unable to determine what computations are required.

The situations to which reference has been made involve a great deal more than mere computation. They involve the ability to detect hidden issues, to work them out and to weigh them; the sense to accept judgment until all necessary facts have been taken into account, and the power to make decisions. They constantly surround the adult in his personal, civic, and business life, and in one way or another they enforce their demands upon him. Their attractive features are presented by one's friends, by one's business associates, and by various civic and political organizations. "Shall we buy for cash or lay on the installment plan?" "Shall we carry this kind of insurance or that?" "Shall we leave our savings in the bank or lay a bond?" "Shall we cash this note for this plan of tax revision or for some other plan?" Is it cheaper more convenient, and more satisfactory to rent a house or to buy one? There are many other similar questions come from the adult - or rather, are presented through various channels for his consideration - and often their attractive features are emphasized and played up in the presentation. The adult must be able, while his attention is caught by the attractive features, to recognize that a given situation presents also certain other, less attractive features and he must seek them out and weigh all together. In short the adult must understand the situations in which he is so intelligently discriminative about their value.

So, if the pupil is to be introduced in the ways of the larger world into which he must eventually enter, he must be led to consider more than certain rule-of-thumb methods and

ing the answers to particular questions, he must be led to study the situations that his previous training has prepared him to study.

If the topic is insurance, for example, the pupil must be led to study the topic, not as a matter solely of number operations, but as a matter of human interest and of intense personal concern as well. What insurance is, why one insures his life or his property, the advantages and obligations of insurance, insurance as a business, insurance as a matter of mutual benefit, both to the company and to the insured, the kinds of insurance and the purposes of each, policies, premiums, and the like are all topics for study and discussion until the personal, human, and business aspects of insurance are understood as well as the ordinary individual who buys insurance needs to understand them. The procedure needed is not merely one of acquiring information; the pupil must gain a view of insurance as a great coöperative enterprise in which he is reimbursed for his losses in turn for his share of the expenses of reimbursing others for their losses. He must gain the view of insurance as a business arrangement of individuals through the medium of a company of qualified officers and directors, in which the obligations and benefits of all are pretty evenly balanced.

Similarly, if the topic is interest, the pupil must become informed of the needs of individuals and of business firms for the use of ready money, of the risks of lenders, of the obligations of borrowers, of the advantages the borrowers gain, of provisions for safety and security, of the similarity between using another's money and using another's property, of the methods of reimbursing the lender for the use of his money, and of like matters. How interest is computed is a matter of secondary concern; but what interest is, why it is charged and paid, the situation that involves it, and the people involved in it are the matters of first importance.

Again, if the topic is taxation, the pupil will need to develop a view of the government as a money-spending organ-

situations is in the main a later development, the pupil need not be compelled to wait. He is at the moment in possession of an idea of number relations that will enable him, as he moves along, to bring the various situations together into certain understandable relations. The various situations may be held together in thought, for the time being at least, and until later studies provide better methods, by the method of referring each situation to the appropriate one of the 'three kinds of problems.' The common idea of percentage runs through all the situations to which we have been referring, and this common idea may be made to stand out with sufficient clearness, when each situation is studied, to bring them all together in thought. With the common idea in clear consciousness, the pupil may observe how the individual, in one situation after another, employs the common idea as a means of handling his personal, business, and civic affairs. For example, the pupil may view the use of one aspect of the common idea in the situation of paying for the use of money, the use of a second aspect in determining the value of an investment, and the use of a third aspect in the manner of accounting profit as a percent of the selling price. As the pupil proceeds, he gains new illustrations of his general idea of percentage, and his general idea suggests a method of attack upon each situation as he comes to it.

IV. ILLUSTRATIVE LESSONS ON APPLICATIONS OF PERCENTAGE

Our exposition emphasizes two points to be observed in guiding pupils in the study of the applications of percentage:

First, lead the pupils to study a situation as a matter of importance in its own right. Help them to view the situation from various angles. Let them observe the situation as an affair of life, as a matter of human interest and of personal, business, or civic concern.

Second, help the pupils to employ their idea of percentage as much as possible in interpreting each situation by itself

and to bring the various situations under the commonest form of the idea of number relations that arise through them all.

The first point is illustrated in the following suggested method of presenting the subject of interest. The discussion of interest there is by no means complete. Although so presented, however, to call attention to the fact that the subject of interest in arithmetic involves a great deal more than mastering certain methods of computation. The material may give some hint also of the manner of helping the pupil to refer to his well-known 'three kinds of problems' in the study of a situation.

A Lesson on Renting Property

When a person needs the use of something he does not own and it is not convenient or advisable for him to buy it, he may rent the use of it from someone who has it to rent. When one arranges to rent a thing, he agrees to pay for its use. Rooms, apartments, storerooms, offices, cars, garages, and many other things are rented by persons who do not own them, but want to use them. The money they pay for the use of a thing is called rent.

The amount of rent that is charged is determined in part by the value of the property that is rented.

1. Mr. Law, Mr. Black, and Mr. Hargus live side by side in rented houses on a certain street in our town. Mr. Law lives in a 6-room frame house and pays \$40 a month rent. Mr. Black lives in a 6-room brick house and pays \$50 a month rent. Mr. Hargus lives in a larger 7-room brick house and pays \$65 a month rent. Why are the three men charged different amounts for rent?

2. Mr. Greenberg, who owns the I. H. Automobile Company, rents the use of cars by the day to anyone who may want them. Among the cars he rents is a second-hand one that cost \$200, one that he bought new for \$675, and a new expensive one that cost \$800. Do you think that he charges the same rent for each of these three cars? Why?

Find out the different rents that are charged for houses, stores, garages, etc., in your neighborhood. How do you can tell why different rents are charged.

The amount of rent that is charged is determined in part by the risk the owner takes when he rents the property.

1. Mr. Greenberg charges \$5 a day rent for the automobile that cost \$675. On the roof of the lot where he lives is a double frame garage that cost about \$700 to build. For the two stalls in this garage he gets \$4 a month rent, which is about 27¢ a day. Why does he charge so much more rent for the car?

2. Near the Iowa Iron Works is a small one-story frame building. Mr. Brown wants to rent it to use as a restaurant, and Mr. Kennedy wants to rent it for a small grocery store. If Mr. Brown gets it, he will, of course, fit up the back part as a kitchen and put in a large stove for cooking. The owner will rent it to Mr. Brown for \$40 a month, or to Mr. Kennedy for \$35 a month. Do you think the owner is doing right to ask different rents for the same building? Why?

3. Mr. Wilson owned a vacant house that he was trying to rent. Mr. Good and Mr. Blank both had looked the house over, but neither had decided to ask to rent it. Mr. Good had the reputation of taking good care of the houses where he had lived and of paying his rent promptly. Mr. Blank had the reputation of being careless about the way he looked after the houses where he had lived and of not being prompt in paying his rent. Mr. Wilson decided if Mr. Good wanted the house, to charge \$30 a month rent, but if Mr. Blank wanted it, to charge \$40 a month. Why?

What things determine the amount of rent one should receive? Do not forget them. The value of the property and the risk the owner takes in renting it.

A Lesson on Renting Money

When a person works and makes money and saves part of it, he can use what he has saved to buy property of some kind that he can either use himself or rent to someone else. Or, if he does not buy property with the money he has saved, he can keep the money for his own use. But, if he does not have any use for his money at the time, he can 'rent' it to someone else who may need to use it. The one to whom money is rented pays for the use of it just the same as a person to whom a house is rented pays for the use of the house.

Money that is paid for the use of property is called *rent*.

Money that is paid for the use of money is called *interest*.

The amount of interest that is charged is determined in part by the rate (amount) of the money that is loaned or borrowed.

1. Mr. Wilson borrowed \$200 from the Commercial Bank. At the end of one year he paid back \$210. The Commercial Bank charged him for the use of the \$200 for one year at 5% interest.

2. Henry borrowed \$100 from the Bank of the City. He paid back \$105 at the end of one year. The Bank charged him for the use of the \$100 for one year at 5% interest.

3. A business firm had to borrow \$10,000 to carry on its business. The firm had to pay \$1,000 a year interest for the use of the money.

4. At the end of a year the firm paid back \$10,000 of the \$10,000 loan. For the use of the \$10,000 for one year the interest charged was \$1,000.

The amount of interest that is charged is determined in part by the length of time the money is used when borrowed.

1. On several occasions the Atlantic Telephone Company has had to borrow money for use in its business. At one time the company borrowed \$10,000 for 1 year. The interest that had to be paid was \$1,000. At another time the company borrowed \$10,000 for 6 months. The interest that had to be paid was \$500. At another time the company borrowed \$10,000 for 3 months. The interest that had to be paid was \$250. At another time the company borrowed \$10,000 for 1 month. The interest that had to be paid was \$50.

2. If the interest that is charged for \$10,000 for 1 year is \$1,000, how much interest should be charged for 2 years? For 3 years? For 1½ years? For 6 months? For 3 months? For 1 month?

3. If the rent for a house is \$10,000 for 1 year, how much should be paid for 2 years? For 3 years? For 1½ years? For 6 months? For 3 months? For 1 month?

A Lesson on the Rate of Interest

The amount of interest that is charged is determined in part by the rate of interest.

Rate of interest means the percentage of the money borrowed that is loaned, or borrowed, for one year. It is usually expressed, as, 5%

described, as "interest at so many percent." Thus, if the rate of interest is 6% for 1 year, the rate is spoken of, as described, as "interest at 6%," or "at 6% interest." Interest at $3\frac{1}{2}\%$ means a rate of $3\frac{1}{2}\%$ a year. Interest at 3 means a rate of 3% a year, and so on.

1. Kelton Farley needed \$300 to help pay his expenses during his last year in college. He learned that the college would lend him the money from the Students' Loan Fund at 6% interest. He thought that it might be 2 or 3 years before he could pay back the money. How much interest would he have to pay each year?

This is the way to answer the question:

\$300	The amount he wanted to borrow
.06	The rate of interest
<hr/> \$18.00	The amount of interest for 1 year

2. Suppose Kelton had borrowed the money from the Students' Loan Fund, and had paid it back at the end of $2\frac{1}{2}$ years. How much interest would he have had to pay for that length of time?

\$300	The amount borrowed
.06	The rate of interest
<hr/> \$18.00	The amount of interest for 1 year
$2\frac{1}{2}$	The time the money was borrowed
<hr/> 9.00	
<hr/> \$27.00	
<hr/> \$45.00	The amount of interest for $2\frac{1}{2}$ years

3. Kelton's uncle learned that Kelton needed some money and offered to lend him the \$300 at 4% interest. Kelton decided to borrow the money from his uncle. He paid back the money at the end of 2 years. How much interest did he have to pay?

Notice how the problem is worked:

\$300
.04
<hr/> \$12.00
2
<hr/> \$24.00

Remember that "at 6% interest," "or interest at 6%" means 6% for 1 year; "at 3% interest" means 3% for 1 year; and so on.

4. A college student of Madison's family, aged 21, borrowed and kept the money for 3½ years. How much interest did Madison's share have to pay?

(1)

(2)

Find the interest for 1 year. Find the interest for the whole time.

\$250	\$42.50
64	34
\$42.50	21 25
	127 50
	\$146 75

Madison's share had to pay \$146 75 interest for the sum of \$250 for 3½ years.

- (1) Find the interest for 1 year by multiplying by the percent.
- (2) Find the interest for the whole time by multiplying by the number of years.

A Lesson on Short Loans

Usually, when money is borrowed for personal or business use it is borrowed for periods shorter than a year — sometimes for 30 days (1 month), sometimes for 60 days (2 months), sometimes for 90 days (3 months), sometimes for 3, 6 or 9 months, and so on.

1. Mr. Brown, the grocer, had just bought a supply of groceries. By paying for them within 30 days he can get them a little cheaper. But his customers will not let him pay them until before the end of the month, and he does not have enough money now to pay for the supply of groceries. He is very sorry because he has to pay cash for his supply of groceries, and that he must get them a little cheaper than usual. He wishes he got his supply of groceries now, he may borrow money for 30 days, and by then time his customers will have paid him enough to let him pay back what he needs to borrow.

2. Mr. Jenkins owns an orchard. In August he wishes to buy some barrels and baskets for the fruit he will pick and pack. During the early fall he will have to have several baskets for picking and packing the fruit. By November, he expects to have the fruit, at least most of it, marketed. But he does not have enough ready money in August to pay for the barrels and baskets and for the help he must have, and he does not expect to collect much from

for the fruit he will sell until the middle of October. He may borrow enough money in August to be paid back in 3 months, to see him through.

3. Mr. Warth, who is a contractor and builder, tries to be prompt about paying for the materials he has to buy, and he has to pay the men who work for him regularly each week. He is building a school for the Board of Education. He knows that the Board will not pay for the remainder they will owe him, until the building is finished ten months from now. But he needs \$12,500 now to pay his bills for materials and labor. What can he do? He may borrow \$12,500 for 3 months, and by that time he can pay back what he has to borrow with part of the money the Board of Education will pay him for finishing the work on the schoolhouse.

4. Mr. Hall had some extra household expenses, so he borrowed some money from his insurance company. He then was able to pay for his extra household expenses at once. He was able at the end of 8 months to pay back the money he had to borrow.

5. Mr. Brown borrowed \$750 for 1 month at 6% interest. How much did the interest cost him?

	\$1 75	interest for $\frac{1}{12}$ of a
	12) \$45.00	year, or 1 month
\$750	75	
.06	90	
\$45.00 interest for 1 year	84	
	80	
	60	

6. Mr. Jenkins borrowed \$1250 for 3 months at 6% interest. How much interest did he have to pay?

	\$16 75	interest for $\frac{1}{4}$ of a
	4) \$75.00	year, or 3 months
\$1250	4	
.06	25	
\$75.00 interest for 1 year	32	
	30	
	28	
	20	
	20	

2. Willard Crum delivers papers and sells magazines in our neighborhood, and deposits most of the money he makes in a savings account at the bank. The bank takes care of Willard's money for him, and besides pays him 3% interest for the use of it. The bank pays the interest each January 1 and each July 1. Since the interest the bank pays is 3% a year, the interest it pays for 6 months is $1\frac{1}{2}\%$.

This is a fine way for Willard to save and invest his money. He is sure of getting it when he needs it, and he is paid for the use of it.

2. In order to get money when it is needed for war, states, counties, cities, school districts, railroads, large business firms, and other organizations will borrow that money in pay various amounts of interest. For example, a state will sell bonds to get money to build roads, a county to get money to build a bridge, a city to pay for street paving, a school district to pay for a new building, a railroad to buy new engines and cars, a large business firm to build a new factory, etc.

Mr. Johnson had \$1200 to invest. He decided to buy bonds. At the time he could buy bonds from his state that paid 4% interest, or from a local manufacturing company which paid $3\frac{1}{2}\%$ interest. Which is better - 4% of \$1200, or $3\frac{1}{2}\%$ of \$1200? Mr. Johnson decided to buy the 4% bonds from his state. Why?

4. Sometimes people buy stock as an investment. Stock means a share in the business. When a person buys stock, or a share in a business, he gets his share of all the company makes. If the company makes much money, his share of the profits is large. If the company makes only a little, his share amount is small. If the company does not make any profit, the one who has bought stock makes none. If the company fails, the owner of the stock loses what he has invested.

Stock in a reliable, successful company is a good investment, but it is not easy to tell just how reliable a company is or just how successful it will be in the future. It is not advisable for a person who knows but little about business to buy stock.

a. Mr. Wheaton bought \$200 worth of stock in a local oil and gas company. For $2\frac{1}{2}$ years the company paid a dividend of 30%. (Dividend means a share of the profits, it is the same as interest for the use of money invested. Do not confuse with the dividend one speaks of in division.) How much is 30% of \$200 for $2\frac{1}{2}$

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years? This was a fine investment, was it not? But at the end of 2½ years, the company failed, and Mr. Henderson lost the \$200 he had invested. Is it or wasn't?

b. Mr. Henderson bought \$2000 worth of stock in a small new company. The dividend is declared on the par value at 8%. Mr. Henderson received \$160 for a year. During the fourth and fifth years, he received no dividend, because the company was not making any profits. The sixth year the company failed.

c. Mr. Henderson bought \$2000 worth of stock in a very large company that does business over the whole country. For three years his dividend was 8½%. For two years it was 7½%, and now he is getting a dividend each year of 6½%. The company has a growing business, and it looks as though the dividends in the future will be larger than 6½%. Therefore the company seems to be such a reliable one. Mr. Henderson could now sell the stock that cost him \$2000 for more than the value of the company he owned for \$2750.

A Lesson on Finding How Much Interest

Remembering his investing in investments a person knows how much it will cost him and how much he will receive in all any given month, year, and he knows in how long time each dividend he would get on his money if he should make the investment.

1. You remember that Mr. Henderson, who bought \$2000 worth of stock in a good company, was finally getting 8½% dividend on his investment. Each year Mr. Henderson was getting 8½% of \$2000, or \$170. Let us suppose that Mr. Smith buys Mr. Henderson's stock from him for \$2750. Mr. Smith will then be getting the same amount of dividend each year from the company for the same share of stock. \$170. (Do not think that Mr. Smith buys Mr. Henderson's share of stock cheap and that anything to do with the amount of business the company is doing, and the company of course, does not raise the dividend just because Mr. Smith pays Mr. Henderson more for the stock than Mr. Henderson paid for it. Mr. Smith has to pay \$2750 to get a dividend of \$170 and he would like to know just how much interest \$170 is on \$2750. In other words, he would like to know what percent \$170 is of \$2750.)

(When we want to find what percent one quantity is of another, we put the number which stands as the dividend.)

This is the way he finds out

$$\begin{array}{r} 041, \text{ or } 4\% \\ \$2700 \div \$170000 \\ 10000 \\ 11000 \\ 11000 \end{array}$$

So, by buying the stock for \$2700 that pays a dividend of \$170, Mr. Smith will be making 4.4% interest on his investment.

2. Mr. Jones reads in the paper that he can buy stock in a certain company for \$100 a share, and that he would be paid \$9 a share in dividends each year by the company. He wants to know what percent interest that would be. In other words, he wants to know what percent \$9 is of \$100.

$$\begin{array}{r} 09, \text{ or } 9\% \\ \$100 \div \$9000 \\ 000 \end{array}$$

3. Mr. Johnson paid \$1200 for a small share in a certain business. Each year he is paid about \$60 as his share of the profits. What percent of interest does he get on his investment?

4. Mr. Davis reads in his newspaper the prices that are being charged for shares of stock in various companies, and the amount of dividend each different share is paying. He makes a little table like the one shown below. He labels each company A, B, and C, and so on, and opposite each he writes the cost of a share of stock and the amount of the dividend that is paid. He next finds the percent of interest each of the different shares will pay. Find the percents for him. (Do not carry the division beyond tenths of percent. Notice how A is worked.)

$$\begin{array}{r} 0422 = 4.22\%, \text{ or } 4.4\% \\ \$57 \div \$25000 \\ 225 \\ 220 \\ 171 \\ 400 \\ 456 \end{array}$$

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Company	For each share	For each share	For each share
1.			
2.			
3.			
4.			
5.			
6.			
7.			
8.			
9.			
10.			

6. Suppose all of the companies are good, reliable companies. Which one offers the best investment? Why?

8. Suppose Mr. Blake had \$1250 to invest. If on every share of stock can be buy in the company that offers the best investment?

7. Suppose Mr. Blake does not know anything about any of the companies. Should he invest in any one of them? Why?

8. Suppose Mr. Blake knows that 4 companies A, B, C, and D are good, reliable companies and that he cannot go into about the rest. Which company offers the best investment? Why? If on every share of stock can be buy with his \$1250 in the company?

A Lesson on the Value of an Investment

What a thing is worth is not always the same as what it costs. Sometimes a thing is worth more than it costs, sometimes it is worth less, but usually the value and the cost are just about the same.

The value, or the worth, of a thing is shown by the service it gives. The cost of a thing is the cost of making that service and gives good service is worth more than the cost that service cost. quickly even though the price may be too high. The value of a thing is that given the greatest service has the greatest value. If two investments are equally safe, the one that brings the highest per cent of interest has the highest value.

Often a person is mistaken about the value of a thing because the cost of it is high. Mr. Blake had been buying a certain kind of stock for \$25.00 a share and was very well satisfied with them. Last fall he liked the looks of another kind of stock and was buying some

for \$20.00 a pair. He thought he would try a pair of them. Since they cost more than the ones he had been wearing, he thought they surely were worth more. When he brought a pair and wore them, he found that they were no larger and gave no better service than the \$15.00 kind he had been wearing. Did he find that the \$20.00 shoes were worth more than the \$15.00 shoes? Why?

Henry Smith is a traveling salesman. He drives his car to call on his customers in different towns in the state, and he keeps account of all of his expenses. He has used three kinds of tires on his car. One kind cost him \$5.00 each, another cost him \$6.00 each, and a third kind cost him \$8.00 each. The following table shows how he compared the costs and the values of the three kinds of tires:

KIND OF TIRE		
First Kind	Second Kind	Third Kind
Cost of each tire	\$5.00	\$6.00
Cost of 100 tires	\$500.00	\$600.00
Cost of 200 tires	\$1000.00	\$1200.00
Cost of 300 tires	\$1500.00	\$1800.00
Cost of 400 tires	\$2000.00	\$2400.00
Cost of 500 tires	\$2500.00	\$3000.00
Cost of 600 tires	\$3000.00	\$3600.00
Cost of 700 tires	\$3500.00	\$4200.00
Cost of 800 tires	\$4000.00	\$4800.00
Cost of 900 tires	\$4500.00	\$5400.00
Cost of 1000 tires	\$5000.00	\$6000.00

Which kind of tire was worth least?

Which kind had the greatest value?

Does the cost of a thing give any idea of its value?

Does the cost tell exactly what the value is?

Sometimes when a person tries to buy a thing the owner will ask a price that is more than the thing is worth. The person who wants to do the buying has to keep in mind that the price of a thing does not always tell exactly what the value is. Sometimes when a person has money to invest, the owner of the property, or of the stock, or of the bonds will ask a price that is more than the value. The person who has money to invest must remember that the price of an investment does not always tell exactly what the value of the investment is.

How does a person tell the value of an investment? He must think not so much about the cost of it as about the service it ought to give. Now, as you know, the service that an investment gives is the interest, or dividends, it brings the owner each year. If this service is less than it ought to be, the value of the investment is

lower than the price of 4. If the answer is more than one should reasonably expect the value to be higher than the price. The question that one has to ask is: How much interest, on dividends, should one reasonably expect when he invests? Or, what percent of interest, or dividends is reasonable?

1. Mr. Brown knows of an investment that pays \$75 a year in dividends. He believes that this kind of an investment ought to pay 6%. In other words, he thinks that the dividend of \$75 is 6% of the value of the investment. What is the value of this investment?

Notice: When a percent (6%) is given of a number (value of investment) that is not given, divide by the percent.

$$\begin{array}{r} \$1250 \\ 6\% \overline{) \$75.00} \end{array}$$

6 Mr. Brown figures that this investment
15 would be worth \$1250 to him
12
20
30
30
0

2. Mr. Hendrickson is thinking about buying a house for rent. That is, he wishes to buy a house as an investment. He believes that the rent one gets each year should be 10% of what the house is worth. The house he is thinking of buying rents for \$120 a month, or \$420 a year. How much does Mr. Hendrickson figure he can pay for the house?

According to the way Mr. Hendrickson figures it, the \$420 yearly rent is 10% of the value of the house or of what he can afford to pay. Thus, he figures that the value of the house is \$4200, and that he can afford to pay \$4200 for it, and so more

$$\begin{array}{r} \$4200 \\ 10\% \overline{) \$420.00} \\ 40 \\ 20 \\ 20 \\ 00 \end{array}$$

3. Mr. Jones reads in his paper that he can buy stock in a certain well-known company for \$105 a share. He is interested, not only in what the stock costs a share, but also in what it is worth a share. He knows that the company pays \$8 a share dividends each year. Mr. Jones figures that he should get 5% interest for money invested in the stock of the company. What is a share of this stock worth to Mr. Jones?

Pay attention, not to the cost, but to the value.

Notice: The \$8 a share is 5% of what Mr. Jones thinks the stock is worth to him. He figures that the stock is worth \$160 a share, and that the price of \$105 a share is low high.

$$\begin{array}{r}
 \$160 \\
 25\% \text{ of } \$160 \\
 \hline
 40 \\
 160 \\
 \hline
 200
 \end{array}$$

4. Below are given the costs and the yearly dividends of some investments. Figuring the yearly dividend on each case as 5% of the value of the investment, find the values of the investments. Pay no attention to the cost in finding the value. Pay attention to the service in finding the value. When you have found the value of an investment, compare it with the cost, and tell whether or not the investment is a bargain.

Cost of Investment Yearly Dividend Value of Investment

a. \$105 a share	\$8.00 a share	?	?
b. \$43 a share	\$3.00 a share	?	?
c. \$195 a share	\$5.00 a share	?	?
d. \$50 a share	\$4.50 a share	?	?

5. Copy the costs and the yearly dividends in another table, and find the answers to the questions if the dividend in each case is thought to be 5% of the value.

6. Find the amount of the dividend in each case in the light of the 7% of the value.

7. A small apartment house is for sale, and can be bought for \$30,000. The six families who live in the apartment pay rent that amounts to \$2000 a year. Figuring the yearly rent as 10% of the value of the apartment, what is the value? Is your result that the price of \$30,000 is too high?

V ILLUSTRATE LESSONS ON THE THREE KINDS OF PROBLEMS

How pupils may be aided in employing their ideas of the three kinds of problems is illustrated in the following suggested material. It will be observed that pupils have to learn something more than formulas of procedure. The material serves also to illustrate the fact that the situations must be studied until the ability to make distinctions, or to choose between various possible modes of procedure, has been developed.

Explaining the Use of Formulas in the Three Kinds of Problems

Sometimes a person knows the cost of a thing, and the percent of gain or loss, and wants to find the selling price. He has known the former number, and the percent of increase or decrease, and wants to find the percent number. In such a case he has a 'problem of the first kind.'

Sometimes he knows the cost and the selling price, and wants to find the percent of gain or loss. He has known the former number and the percent number, and wants to find the percent of increase or decrease. In such a case he has a 'problem of the second kind.'

Sometimes he knows the selling price and the percent of gain or loss, and wants to find the cost. He has known the percent number, and the percent of increase or decrease, and wants to find the former number. In such a case he has a 'problem of the third kind.'

There are just three kinds of problems in gain or loss in which percent are used. These 'three kinds' are the same three kinds that you learned to work with, as fractions, decimals, and percents. Remember how each is worked.

First kind: When a percent is given of a number that is given, multiply by the percent.

Second kind: When two numbers are given and you are asked to find the percent, place the number being asked about in the dividend.

Third kind: When a percent is given of a number that is not given, divide by the percent.

A Lesson on Problems of the First Kind

1. A druggist bought some hot-water bottles at a cost of \$1.20 each. He sold them so as to gain 25%. At what price did he sell them?

This is a two-step problem. First we must find how much gain the druggist made on each bottle, then we must add this gain to the cost of each to find the price at which each was sold. The "25% gain" means 25% of the cost.

\$1.20	\$1.20 cost
.30	.30 gain
<hr/> 1.50	<hr/> \$2.40 selling price
.50	
<hr/> \$2.00 gain	

Or we can think and solve the problem this way: Cost is 100% of the cost; gain is 25% of the cost. Cost and gain added give the selling price. So the selling price is 125% of the cost.

\$1.20	100%
.30 (1.25 is the same as 125%)	25%
<hr/> 1.50	<hr/> 125%
.50	
<hr/> \$2.00	
1.20	This is the 'short way.' Think, "Gain of 25%
<hr/> \$2.40	makes selling price 125% of cost."

2. The Horace Mann School was built to take care of 500 pupils. Last summer it was enlarged so that it could take care of 25% more pupils. How many pupils will it now take care of?

"25% more" means an increase of 25% of 500.

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500	500 lowest number
25	125 increase
25 00	625 percent number
100 0	
125 00	

Or we may solve the problem the 'short way' by thinking, "25% more makes the percent number 125% of the lowest number," and finding 125% of 500

500
1 25
25 00
100 0
500
625 00

3. After the fire at the clothing store a quantity of clothing that cost \$540 was sold at a loss of 6%. How much did the store get for the clothing?

This is a two-step problem. First we must find how much the clothing store lost on the sale of goods. Then we must subtract the amount that was lost from the cost to find the selling price. Loss of 6% means 6% of the cost.

\$540	\$540 cost
63	6% loss
47 00	\$594 selling price
504 0	
\$540 00	Cost - Loss = Selling Price

Or we can think and solve the problem this way. Cost is 100% of the cost, loss is 6% of the cost. The loss subtracted from the cost gives the selling price. So the selling price is 94% of the cost.

\$540	100%
63	6%
47 00	94%
504 0	
\$594.00	This is the short way. Think, "Loss of 6% makes the selling price 94% of the cost."

4. The population of a certain town was 3240 in 1920. In 1930 the population had decreased 25%. What was the population of the town in 1930?

The decrease of 25% means 25% of the former number

A Lesson on Problems of the Normal Kind

1. A grocer bought apples at \$1.00 a bushel and sold them at \$2.00 a bushel. What percent of the cost did he gain?

In this problem we have two things to do. First we must find how much was gained on a bushel, then we must find what percent this gain is of the cost of a bushel.

The first question is, "How much was the gain?" This is the way we find it.

$$\text{Selling price} - \text{Cost} = \text{Gain}$$

$$\begin{array}{r} \$2.00 \text{ selling price} \\ 1.00 \text{ cost} \\ \hline \$1.00 \text{ gain} \end{array}$$

The next question is, "What percent of the cost (\$1.00) is the gain (\$.40)?"

$$\begin{array}{r} 25, \text{ or } 25\% \\ 1.00 \overline{) 40.00} \\ \underline{32 \ 0} \\ 8 \ 00 \\ \underline{8 \ 00} \\ 0 \end{array}$$

Or we can solve the problem this way. First find what percent the selling price is of the cost, and then subtract 100% of the cost to find the percent gained.

$$\begin{array}{r} 1.25, \text{ or } 125\% \\ 1.00 \overline{) 2.00} \\ \underline{1.00} \\ 1.00 \\ \underline{1.00} \\ 0 \end{array}$$

Thus we find that the selling price is 125% of the cost. The cost, we know, is 100% of itself. So the gain is

$$\begin{array}{r} \text{Selling price} - \text{Cost} = \text{Gain} \\ 125\% - 100\% = 25\% \end{array}$$

Which of the two ways of solving the problem is the easier?

1. INTEREST ON THE TRUCK RATES OF FREIGHTS 4.32

2. Last year the consideration of the freight rates was 4.32. This year the consideration is 4.32. It has to be proved that the freight rates are the same as last year.

4.32 12% of 12 1/2%
 4.32 4.32 54 0000
 24 000000 4.32 The freight rate is 12 1/2% of last year's consideration (4.32)
 30 000 4.32
 4 32
 2 1400
 2 1400

1 12%, or 12 1/2%

4.32 12% of 12 1/2%
 4.32 There are four that this year's consideration (4.32) is 12 1/2% of last year's consideration (4.32)
 54 0 12 1/2% of last year's consideration (4.32)
 4.32 12 1/2% of last year's consideration (4.32)
 10 000 12 1/2% of last year's consideration (4.32)
 5 04 12 1/2% of last year's consideration (4.32)
 2 1400
 2 1400

Which way is the correct?

3. In the last year of the consideration the freight rates were fixed a number of times that was 4.32. The freight rates are a constant rate and will remain the same for 4.32. It has to be proved that the freight rates are the same as last year.

In the present year there are 4.32 of 12 1/2% of last year's consideration (4.32). It has to be proved that the freight rates are the same as last year.

The first question is, "What is the freight rate?" It has to be proved that the freight rates are the same as last year.

4.32 12% of 12 1/2%
 4.32 4.32 54 0000
 24 000000 4.32
 30 000 4.32
 4 32
 2 1400
 2 1400

The second question is, "What is the freight rate?" It has to be proved that the freight rates are the same as last year.

40 or 40%

$$\begin{array}{r} \$56 \overline{) \$23.20} \\ 224 \\ \hline 80 \\ 80 \\ \hline 0 \end{array}$$

Or we can solve the problem this way First find what percent the selling price is of the cost, and then subtract this percent from 100% of the cost to find the percent loss

40, or 40%

$$\begin{array}{r} \$56 \overline{) \$24.00} \\ 224 \\ \hline 160 \\ 168 \\ \hline 0 \end{array}$$

Thus we find that the selling price is 40% of the cost. The cost, we know is 100% of itself, so the loss is

$$\begin{array}{rcl} \text{Cost} - \text{Selling price} & = & \text{Loss} \\ 100\% - 40\% & = & 40\% \end{array}$$

Which way of solving the problem is the easier?

4. Last year Mr. Johnson raised 1072 bushels of corn. On account of the dry weather this year he raised only 670 bushels. What was the percent of decrease?

Notice the two ways of solving the problem

$$\begin{array}{r} 1072 \overline{) 670} \\ 670 \\ \hline 402 \text{ decrease} \end{array}$$
$$\begin{array}{r} 1072 \overline{) 402.000} \\ 3216 \\ \hline 8040 \\ 7504 \\ \hline 5360 \\ 5360 \\ \hline 0 \end{array}$$

62.5, or 62.5%

$$\begin{array}{r} 1072 \overline{) 670.000} \\ 6432 \\ \hline 2680 \\ 2144 \\ \hline 5360 \\ 5360 \\ \hline 0 \end{array}$$
$$\begin{array}{r} 100.0\% \\ 62.5 \\ \hline 37.5\% \end{array}$$

Which way is the easier?

In working this problem we must keep in mind that the word 'loss' is an important one. The 20% loss is not 20% of \$1284, but is 20% of the cost of the cattle that were sold. The selling price, \$1284, is not 20% of the cost. The problem states that the selling price, \$1284, made the sale a loss. We must keep in mind that the selling price is the most loss the merchant lost. That is, 20% of the cost subtracted from the cost, which is 100% - 20%, or 80% of the cost. Now, since we know that \$1284 is 80% of the cost, which is not given, we can find the cost by dividing by the percent.

We think, "Loss of 20% makes selling price 80% of the cost," and solve as follows.

$$\begin{array}{r}
 \$1284 \\
 .80 \overline{) \$1284.00} \\
 \underline{800} \\
 484 \\
 \underline{400} \\
 8400 \\
 \underline{8000} \\
 4000
 \end{array}$$

4. There are 49 members in the high-school band this year. This is a decrease of 12.5% of the number of members in the band last year. How many members were in the band last year?

Think, "Decrease of 12.5% makes the present number 87.5% of the former number."

$$\begin{array}{r}
 56 \\
 .875 \overline{) 49.000} \\
 \underline{4375} \\
 5250 \\
 \underline{5250} \\
 0
 \end{array}$$

A Lesson on Figuring Profit on the Selling Price

In the problems you have just been having, the percent of profit was figured on the cost of the goods that were sold. This is the usual way of figuring the profit. Often, however, a merchant or a business firm wishes to figure the profit as a certain percent of the selling price. There is a perfectly good reason for figuring the profit this way. The reason is that at the end of a day's sales or

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at the end of a week or a month or for that matter the business man knows just how much money has been taken in for the week or what percent of the money taken in he can sell his goods. For example, suppose the cash register at the grocery store shows at the end of the day that \$175.00 is the amount that has been received for the groceries sold during the day. If the grocer has his business well planned that he can sell 25% of the goods he has on hand at once just after he has received the goods he would have of course, in this case that \$43.75 of the \$175.00 as profit. If at the week's end he is selling all his merchandise at 25% off the top, and the grocer knows that his cash register has received \$175.00 at the end, he can tell at once just how much he has received for the week.

So business of the sort of buying and selling goods just from month to month or year to year is the sort of business that is most common and business men who do this sort of business know that their cash register shows profits as a constant percent of the selling price. For this reason they know how much they have for the week or month or year. Let us find out how they do this.

1. The manager of a retail store bought several boxes of \$1.20 each. He wanted to know what price to charge for each box so that he would receive 40% of the selling price. He had to find the charge for each box.

Let us help this manager to solve the problem. We will suppose the problem was not so much the manager's business, but that he wants to find out:

(a) What the manager knows

The cost of each box \$1.20

The percent profit he wants to make 40% of the selling price

(b) What the manager wants to find out

The price to charge the selling price for each box

The problem does not say that the selling price is 40% of the cost, so we cannot find 40% of \$1.20 at the end of the week or any way.

The problem does not say that the cost is 40% of the selling price, so we cannot divide \$1.20 by 40 to get \$3.00 as the selling price.

The problem does say that the profit is to be 40% of the selling price. If we knew the profit we could divide it by .40 to find the selling price. But instead of knowing the profit we know the cost.

Here is one other thing the manager knows:

$$\text{Cost} = \text{Selling price} - \text{Profit}$$

So the manager would think to himself, "The selling price is 100% of itself, and the profit is 40% of the selling price, so the cost must be 60% (100% - 40%) of the selling price. The cost is \$1.20, and this is 60% of the selling price. So the selling price is \$1.20 divided by .60, or \$2.00."

100%	\$2
- 40)
-----	\$1.20
60%	1.20

If, then, we are to help the manager to solve his problem, we need to know that

$$\text{Cost} = \text{Selling price} - \text{Profit}$$

2. When the profit is 40% of the selling price, the cost is 100% - 40%, or 60% of the selling price.

Fill the blanks:

3. When the profit is 35% of the selling price, the cost is . . . % of the selling price.

4. When the profit is 25% of the selling price, the cost is . . . % of the selling price.

5. When the profit is 20% of the selling price, the cost is . . . % of the selling price.

When a person figures his profit as a certain percent of the selling price, he finds what the selling price of his goods ought to be, as follows:

- (1) He subtracts the percent of profit from 100%
- (2) He divides the cost by this percent.

He does these two things because

(a) $\text{Cost} = \text{Selling price} - \text{Profit}$

(b) When a percent is given of a number that is not given, one should divide by the percent.

6. Another grocer bought sugar beets and sold them for \$1.17 a dozen cans. He sold the cans at 10% under a profit of 35% of the selling price. How much did he charge for each dozen cans?

Find the selling price in each of the following cases:

Cost	Profit	Selling price
7. 60¢	25% of the cost	?
8. \$25.00	30% of the selling price	?
9. \$38.00	35% of the cost	?
10. \$1.50	40% of the cost	?
11. \$1.25	45% of the selling price	?
12. \$12.00	20% of the selling price	?

CHAPTER XX

MEASURES AND WEIGHTS

ARTICLE XXV

1 The units and subdivisions of measurement used throughout are those provided by these regulations.

2 Number when the word degree is used measurement for that designated degree of the indicated measure, measurement and weights designated by these regulations.

3 Measurement and weights used in all cases are indicated by the words which indicate the measurement or weight.

4 The words of the following of weight measurement shall be used only by the word and shall be used only by the word.

5 The words of measurement, weight, shall be used only by the word and shall be used only by the word.

6 The words of weight, shall be used only by the word and shall be used only by the word.

7 The word measurement of measurement shall be used only by the word and shall be used only by the word.

8 The words of weight, shall be used only by the word and shall be used only by the word.

9 The words of weight, shall be used only by the word and shall be used only by the word.

(a) Measurement of measurement.

(b) Measurement of measurement.

10 The words of the following shall be used only by the word and shall be used only by the word.

11 The words of weight, shall be used only by the word and shall be used only by the word.

12. A chief difficulty of pupils is explained by the fact that they frequently fail to derive from their study of familiar measures a method of measuring those that are not familiar.

13. Lessons presented as offered in illustrative form pupils may be made recognition of a method of attack upon the problem of measuring, in their study of known measures and of the measurement of rectangles.

14. Further material is offered in illustrative form the use of the method of attack to make intelligible the measurement of triangles, circles, etc.

15. Measurement may serve to serve to the pupil a social institution, a means and a record of man's progress.

16. Measurement will, however, not mean more than the teacher undertakes to have it mean.

THE MEASUREMENT OF LENGTH

Measures and weights may be considered from the standpoints of their place in the curriculum in arithmetic, of their historical development and present characteristics, of the demands they make upon the individual who sets out to understand them, of the difficulties pupils encounter in trying to learn them, of the progress of pupils in dealing with them, and of the influence they exert upon the methods of thinking of the modern world. These various points of view are so intimately interrelated as to make it difficult to give explicit treatment of any one of them. We shall content ourselves with the attempt to touch upon the various points of view and to give attention in passing to their relations, similarities, and mutual interdependencies.

I. THE PLACE OF MEASURES AND WEIGHTS IN THE CURRICULUM

One of the first considerations to take the attention of the teacher who gives thought to measures and weights is their place in the curriculum. Although not logically the first question to be asked about them, yet actually the first one to occur to the teacher, whose major interest is to keep the

stick. Its use is not obvious and what is significant is the outgrowth of a long line of human experiences that are wholly unfamiliar to the child when merely to use the stick. How the yardstick may be used is really controlled by an idea in the mind. It is this idea that is unfamiliar in the beginning, and that must be made to develop before the child can use the yardstick with intelligence or can be referred to its use as an illustration of number relations.

2 Historical Development

Although some form of crude measurement must have been used by primitive peoples from the earliest beginnings of human development, although measurement and number experienced a parallel and concurrent development in the history of human progress, although numbers may have been used from the outset to count the crude measures that were made by early peoples, although measurement and number must have been intimately related from the beginning, owing to the fact that the development of the latter depended upon the development of the latter, the fact remains that number ideas do not depend upon measurement for their development. When one has learned to count, for example, he can count his measures if he has any to count. His development of counting may proceed likewise entirely apart from his development and use of measures. That general discussions, however inadequate and inefficient they may be, may be referred to as a very imperfect illustration of the fact that number ideas may develop independently of ideas of measurement. We have proceeded up to the present chapter in this book, dealing with the development of number ideas in the history of the race and in the progress of the individual child through the elementary school, without once having to refer to the development and use of measures and weights for purposes of illustration.

The teacher who undertakes to illustrate number relations by reference to measures, even the most commonly used

more, however, in the future. The government has not yet decided whether it will continue to provide for the future of the people, and if so, in what way. The government has not yet decided whether it will continue to provide for the future of the people, and if so, in what way.

3. General Principles of the Law of the Future

The principles of the law of the future are not yet fully developed. The government has not yet decided whether it will continue to provide for the future of the people, and if so, in what way. The government has not yet decided whether it will continue to provide for the future of the people, and if so, in what way.

The principles of the law of the future are not yet fully developed. The government has not yet decided whether it will continue to provide for the future of the people, and if so, in what way. The government has not yet decided whether it will continue to provide for the future of the people, and if so, in what way.

familiar, and to understand the logic of attempting to teach new situations before the general ideas that apply to them have been acquired. We do not attempt to illustrate the idea of percentage, for example, by reference to the paying of interest on borrowed money, the buying of loans, the value of an investment, and the like, which are usually unfamiliar situations before the pupil has mastered them, we rather delay the so-called "applications" until the pupil has learned something about percentage. Likewise, we must not attempt to illustrate such number relations as are experienced in multiplications and divisions by reference to measures and weights whose relations are unknown and not at all evident in their appearances. It is necessary that the study of measures and weights be delayed until the pupil has acquired by other means the ideas of number relations that appear in them and that help to make them understandable and usable.

All this does not mean that the study of measures and weights be delayed until the end of the elementary-school period; there is no intention to suggest that measures and weights must be the last chapter in the pupil's arithmetic. It is the intention, rather, to suggest that the study of measures and weights be delayed until the ideas necessary for their study have been acquired. This means that the early stages of measures and weights may be undertaken when the ideas necessary for their study have been gained, and that the later stages must be postponed until the ideas necessary for their study have had a chance to develop.

II. THE DEVELOPMENT OF MEASURES AND WEIGHTS

Our question about the grade placement of measures and weights may be answered in another way if we now turn our attention to the historical development of measures and weights and to the characteristic features of their present state of development.

We find that diversity is the outstanding characteristic of measures and weights in their earlier stages of development,

part that seemed to be suggested as the most convenient instrument -- sometimes the span between thumb and middle finger, sometimes the width of the hand, sometimes the length between elbow and tip of finger, sometimes the extension of the arm, and so on.

There then is a source of diversity in the standards even of linear measure, flowing from the difference of the relations between man and physical nature. It would be as inconvenient and unnatural to the organization of the human body to measure a bow and arrow for instance the head circumference of solitary man, by his foot or pace, as to measure the distance of a day's journey, or a warrior's walk to the hunting ground, by his arm or hand.¹

Thus, we see that the first measures used by solitary man for his own individual and personal use differed among themselves. There is, to be sure, no intertable relation between the hand and the pace, as between the span and the step, as instruments of measure. But even where obvious relations exist, as in measures of time, diversity characterizes the early stages of measurement. Early man was cognizant of the difference between night and day, of the appearance of the new moon, and of the change in the seasons, yet he used these for a long time as separate and unrelated measures of the passage of time. A short time was reckoned in days; a longer time in moons; and a still longer time in so many summers or winters. The relations between the three did not become apparent until society reached the point in its development where records were kept and made the subject of comparison. Even then man was confused in bringing the three measures into their exact relations, as may be witnessed by the facts that the early Jewish calendar varied the length of the year between the extremes of 353 days and of 385 days, that the number of days in the month is variable, and

¹ Adams. *Op. cit.*, p. 7.

a shilling, covered and without appearing at all except 32 shillings on the outside of the coin and 20 pence on the inside as usual, and 12 pence was printed. There is no record of additional evidence to show that the quantity used of weight was a grain, sometimes being sometimes reduced, sometimes added.

Common standards of length were often derived from the person of the chiefest of the group, the foot of the king, for example, being taken as the standard foot for the measurement of length.

Common standards will then be assumed from the person of some distinguished individual and associated circumstances, rather than any law of nature will determine whether identity of proportion will be the character of these standards. If, pursuing the first and original objects of nature, the unit should be assumed as the standard of linear measure for the use of the hand, and the pace for the measure of motion, or linear measure upon earth, there will be two units of long measure; one for the measure of motion and another for the measure of motion. Nor will they be comparable in use, for even without the unit and the pace is an aliquot part of a multiple of the other. But, should the discovery have been made, that the foot is an even or aliquot part of the pace, for the measurement of motion, and of the old and fashion, for the measurement of motion, the foot will be made the common standard measure for both and, therefore, there will be only one standard unit of long measure, and its uniformity will be that of identity.

Some semblance of relationship between the various measures crept slowly into general usage. For example, old English law gives us the following tables that show the carrying of relationship back to the use of the grain in weighing:

* J. C. Brown and L. D. Coffman. *New York Truck Arithmetic* (Row, Peterson and Company, Chicago, 1914), p. 173.
Adams. *Op. cit.*, p. 24.

* Adams. *Op. cit.*, p. 11.

V THE HISTORY OF PAST AND PRESENT MEASUREMENT

Although the regulations of government have succeeded in bringing the various measures and weights into clearly stated relationships, and in enforcing strict observance of the established relationships by all traders, great diversity continues to exist with respect to the way the various measures and weights are understood and used. The diversity may be explained in two ways.

In the first place, the various measures are used for different purposes and in different ways. Distance, for example, is measured in miles, and length in feet, yards, or inches. One does not gauge the height of a room or the width of a street as a part of a mile, nor does he speak of the distance between towns in terms of feet. Moreover, the relation between the mile and the foot is hard to grasp. One may note that there are 5,280 feet in a mile, but the number is not very helpful, since it is too large to be readily understood. The number may be used in computations quite as readily as a smaller one, like 528, or 52 8, but when thought of as 5,280 separate feet, the units are too numerous for ready comprehension. One may have fairly definitely in mind the distance of a mile, and may be able readily to estimate short distances, such as 20 feet, 100 feet, 20 yards, and so on, but when he attempts to measure off in his mind 600 yards, let us say, he is helpless until he remembers, or someone suggests to him, that 600 yards is slightly more than one third of a mile. The point is that the two units, the mile and the foot, have separate uses, and are brought into their relations through computations, and through computations only.

Moreover, one learns to use the pound and the gallon for such diverse purposes that he is usually at a loss in estimating the weight of a gallon of water, milk, preserved fruit, etc., until he brings himself to the point of recalling the rhyme, "A pint's a pound, the world around." The ordinary individual is usually somewhat at a loss when he must select

for choosing which, are, that it permits more definite terminations at both ends than any of the separate limbs, and gives a measure easily handled and carried about the person. By doubling this measure is given the ell or arm, including the hand, and half the width of the body to the middle of the breast, and, by doubling that, the fathom, or extent from the extremity of one middle finger to that of the other, with extended arms, an exact equivalent to the stature of man, or extension from the crown of the head to the sole of the foot. For subdivisions and smaller measures, the span is found equal to half the cubit, the palm to one-third of the span, and the finger to one-fourth of the palm. The cubit is thus, for the measurement of matter, naturally divided into 24 equal parts, with subdivisions of which 2, 3, and 4 are the factors, while, for the measurement of distance, the foot will be found at once equal to one-fifth of the pace, and one-sixth of the fathom.*

VI. LEARNING THE MEASURES AND WEIGHTS

Some idea of the peculiar difficulties that the pupil encounters in learning the various measures and weights has been indicated in the foregoing paragraphs. We note that the task is one of learning measures that in their later development have been brought into exactly defined relations, but that in their common and everyday uses are entirely lacking in relations. We note that the measures to be learned possess both the quality of diversity and the quality of uniformity, and that in their common and everyday uses neither quality appears to bring the other to attention, or, in other words, we note that the pupil may be brought to give attention to both qualities without being helped in his study of the one by what he has learned of the other.

For example, the full meaning of the measures in common use is not derived from the study of their common uses, since their common uses do not impress the idea of exactness. One may ask to purchase a pound of butter, a peck of potatoes,

* Adams. *Op. cit.* p. 2.

on a point of view that as far as the thinking of the individual is concerned for policy in the international sphere in almost the same way as he would be in other countries, but not exactly, understood separately. The people would be more quite as well if he asked the government as well as having, a long of position, as a whole of the people. The authority was not understood from the people's view, indeed, it could be pointed out that the people in the government in the authority way, was would not have to be asked for the policy of the people, even more, even the authority was not understood from the authority well. Moreover, the view of a person, however, as usually without reference to any other and the position the any other view and the authority by the way. It was when the relation between the government and the people's authority character, it may be usually the authority and the people's authority, under the authority and the authority's character and their authority. The view of the authority's authority as another authority's authority as a very important way the fact that the relation between the government and the people's authority is not understood by it.

The point is a completely new manner in the world of the world of the world. The point is a completely new manner in the world of the world of the world. The point is a completely new manner in the world of the world of the world.

In France the use of the authority is a completely new manner, as the point is a completely new manner. The point is a completely new manner in the world of the world of the world. The point is a completely new manner in the world of the world of the world.

The nature of the authority is a completely new manner, as the point is a completely new manner. The point is a completely new manner in the world of the world of the world.

* Cf. M. Wilson. "The authority's authority in the world of the world of the world." *Journal of the World of the World*, 1912, 111-112.

the unit of weight, had later recognized their mistake and changed to the kilogram. But the kilogram does not fit well into customary thinking and measurement in trading. The housewife does not want a kilogram of butter, she wants a pound. The pound is a convenient unit for the French people, compelled by severe penalties to use metric terms, took the one-half kilogram (note the " $\frac{1}{2}$ " phrase, a common fraction) and called it a "pound."

In the rural districts of France, the *arpent*, the old French acre, is still the basis of thinking in land measure, and this after 100 years of competition, and 125 years of exclusive teaching of the metric system in the schools.

In Germany the metric scheme has been official and taught in the schools since 1870. But the penalties have not been severe, so the old customary terms are used freely. The market and store everywhere use the pound. The farmer thinks in *arnen* (*Measures*) and bushels and uses those terms. He also uses feet, inches, and yards, particularly feet, in connection with buildings, land, farm machinery, gates, fences, etc.¹

The following conclusions from an article on units of measurement in industry² by a writer who appears to be interested only in the common uses of measures illustrate the fact that if the only aim of the study of measures and weights is to teach common uses, they might just as well not be included in the course in arithmetic.

As a result of this study, it is proposed that the following tentative suggestions should have consideration by the committee that determines the curriculum in arithmetic:

1. It is not profitable for children in the elementary grades to spend time committing to memory tables of weights and measures.
2. When one understands the commodity, it is not difficult for him to apply the preferred unit of measurement to that commodity.

¹ Wilson. *Op. cit.*, p. 219.

² M. del. Louth. "Units of measurement in industry." *Education*, 32: February, 1922, 315-318.

3. The teaching of addition, subtraction, multiplication, and division of compound denominate numbers has little, if any, value.

4. Reduction ascending and reduction descending have little value in industry, with the exception of the estimating department, whose personnel consists of highly-trained experts.*

The two quotations serve to make impressive the lack of dependence of common usage of weights and measures upon an understanding of their relations. If intelligence about measures is not important and if the ordinary individual is not supposed either to be precise in his own thinking or to understand the precise thinking of the world in which he lives -- that is, if common usage is the only matter of importance -- he will not need to study the relations between measures. And, it may be added, he will not need to study the common usage of measures, since he can pick that up in the workaday world without difficulty.

VII. THE STUDY OF RELATIONS BETWEEN MEASURES

The study of the relations between measures that the pupil is frequently called upon to undertake turns out not to be very impressive. One reason is the lack of any ordered scheme into which the various relations may be arranged as they are being learned. Reference has been made to this deficiency in a preceding topic. Another reason is that when the relations are learned, they are taken as a matter of course and often without attempt to grasp their significance. In other words, the pupil is often called upon, when he learns a table of measures, merely to attend to, and to learn the various expressions of relationship. Thus, for example, he learns the number, $5\frac{1}{2}$, which states the relation between the yard and the rod, and makes use of the number in a variety of computations, without having very clearly in

* Leach *Op. cit.* p. 215

mind just how long the yard is and how long the rod is; or he learns that there are 4 quarts in a gallon without any clear conception of the size of either. Forgoing paragraphs have pointed out that common usage does not lead to any great familiarity with the various units of measure. Consequently, when the relation between any two units that have been viewed only in terms of common usage is stated, the relation fails to add to the meaning of the units, simply because the relation, however accurately it may be expressed, between any two vaguely conceived units is itself without meaning.

The matter may be stated in another way. Ordinary usage of measures, as the preceding topic has indicated, can get along very well without any suggestion of relations; and so, any suggestion of relations that may be imposed becomes just so much extra to remember, because it makes no contribution to common usage, which, in a sense, is self-sufficient. Common usage, moreover, employs the various units as diverse units, and being self-sufficient, does not impress the common user with the deficiencies of diversity. Consequently, the significance of any relation that may be imposed upon the learner is lost because there is no place in his scheme of handling measures into which the relation may be fitted. The significance of the relations between measures can hardly be well understood until one begins to appreciate the shortcomings and peculiarities of diverse measures.

VIII. THE STUDY OF DIVERSITIES

The study of early measures is almost a necessary prelude to the study of present relations. The study, for one thing, makes the pupil conscious of diversity, and, with this consciousness, appreciative of the difficulties of earlier peoples who had nothing but diverse measures to use. The study leads, moreover, to an insight into the crude, but easily understood, efforts of early peoples to bring their various measures into relationship; it makes clear the significance

of the various relations that were eventually developed, and it develops a scheme of thinking, or point of view, about measures that requires an understanding of relations for its completion.

The study of early measures makes two other closely related contributions. First, it serves to bring the various measures now in common use each into clear perspective by demonstrating the various stages of their development. Since the measures used by early man had to be readily available and were, necessarily, parts of his body or objects of his immediate environment, their concreteness is a most striking characteristic; and their concreteness serves to make clear their proportions and to give meaning to those of our present measures that have been derived from them.

Second, the study of early measures serves to bring to consciousness the method of procedure that one has to follow in undertaking to measure the amount or quantity of anything. The pupil may observe that the estimate of amount that is derived from measurement is more accurate than any other estimate because it is derived by comparing the thing to be measured, or about which an estimate of amount or quantity is to be made, with something whose size, or amount, or quantity, is familiar or relatively familiar. The pupil can easily note that the available objects, which were selected in the beginning as units of measure, were also familiar objects; and moreover, that the measurement consisted in applying the unit chosen to the thing to be measured. Thus, the two steps — namely, (1) choosing a measure and (2) applying the measure — may be brought to the level of clear consciousness. If the teacher desires, the steps may be made to stand out as ideas of procedure that the pupil can use in his succeeding studies of measures, and that can receive further demonstration in the succeeding studies. Sooner or later in the pupil's progress, he will have to do some thinking of his own in coming to an understanding of derived measures, like square and cubic measure. If he can come to such task with

a clearly defined method of attack, he may be able to understand the better the things he is called upon to learn.

Reference will be made in the subsequent sections that are presented further on to the development and use of a method of attack upon square measure.

IX. Using the Metric System

When the pupil has studied the diversity of early measures and has seen, by way of striking contrast with the measures of the present, the need for their development toward uniformity, he may continue his studies of relations by viewing a system in which relations are not only obvious but indeed the most characteristic feature. The system to which reference has just been made is the metric system of measures and weights. It is proposed that the metric system be included in the curriculum in arithmetic, not for the value of its content, but for the value of its training.

1. Often Reported as 'Useless'

The metric system, insofar as the curriculum of the elementary school is concerned, is frequently disposed of by the statement that its only use is in the study of science, and that the pupil can learn as much as he needs of it later on when his courses in science begin. It is suggested that when the pupil studies the system in the elementary school, he forgets it, and must study it again in the high school. It is pointed out, and entirely correctly, that the uses of the system can be readily learned in connection with its actual uses in scientific measurement; and, it is added, perhaps not entirely correctly, that unless the pupil studies science he has no need for the system. Wilson¹⁰ has suggested that the system's popularity in Europe is more apparent than real, and that the spread of the system has been accomplished by compulsion

¹⁰ Wilson. *Op. cit.*, 319-324.

and propaganda. The following paragraphs are underlines of his generalizations and of his point of view.

Any metric unit which has been steadily accepted as a close approximation is a customary unit.

Metric units which the said steadily approximate customary units have been accepted slowly and unwillingly, if at all.

Metric units are balanced and quartered exactly as the customary units. Incommodious to use in education and ordinary measurement, but is not more exact with metric units than with customary units.

Thus it gradually becomes clear why resistance of non-metric measures, present in metric Europe. The customary units are more convenient and better suited to trade conditions. They persist because of industrial custom and in spite of compulsion.

The manufacturing and trade of the world are still largely non-metric. It would be singular and senseless, and almost without metric units than without customary ones. We need not ask that the metric units be abandoned or legislated against. We can reasonably ask that metric propaganda cease and that metric units be left to men as have no choice. We should avoid compulsion."

If the value of the metric system were determined only by the usefulness of its units as instruments of measurement, our would be justified in denouncing it, with respect to the arithmetic of the elementary school, in any case at all of the ways indicated above. If we accept as correct the point of view that value is determined by common usage, we may conclude with Wilson that the metric system might just as well be abandoned. There is no advantage in measuring with a meter stick in preference to the yardstick, in denouncing grain and kilogram weights into the balance in preference to ounce and pound weights; or in using a liter bottle as a container of liquids in preference to the quart bottle. One can use the one kind of measure just as well as the other, and we

" Wilson. *Op. cit.* 323-324

before. I urge its retention in all systems of measure; once a system has been decided upon, the use is like the use of any other system. There is no gain in turning to the metric system for information or instruction about the way measures may be used.

2. Positive Values Often Overlooked

Let us, then, return to turn to the metric system for more information about kinds of measures and for suggestions about the ways various measures are used. Let us turn to it, if at all, for values that may not be so apparent in our own system or so readily gained from our own system. In learning to use our own system of measurement, we would seek to be intelligent, not only about the various measures themselves, but also about the relations between them. From the point of view of the teacher who may be in possession of an understanding of relations, there is no purpose in seeking elsewhere for an illustration of relations. The teacher must bear in mind, however, that the pupil is not conscious of the relations between the measures he has to learn, and that the measures in common use furnish within themselves a rather poor means of demonstrating relations or even of bringing relations to the level of consciousness. If the teacher is interested in teaching the relations between measures, he had better seek a means of illustration.

The metric system provides just the kind of illustration of relations that is needed. The one standard unit that has been adopted, for each kind of measure stands out with its name and its name is not lost in its divisions and its multiples, and larger units, even though they are given separate and unrelated names -- the *centimeter*, for example, in measuring short lengths, and the *kilometer* in measuring distances -- are held together by the retention of the same name. The name, *meter*, for instance, is the same in *centimeter* and *kilometer*, and the name is useful in suggesting and in emphasizing relations.

Moreover, the exactness of the relations is both suggested and emphasized by the names of the divisions and multiples of the standard unit. The name *cent* denotes one hundredth, and the name *kilo* denotes one thousand. Compounded with the name of the standard unit, *meter*, these names provide names for smaller and larger units that, though used for widely different purposes, retain their relations with the standard in a way that no one can fail to observe; and the compounded names express no relations of relationship with the standard that impresses itself upon the attention.

The metric system also serves to illustrate in a most impressive way the relations between the measures of length and the measures of weight and capacity. The latter measures were derived from the measure of length; they did not grow up as separate measures to be brought finally into relations with the measure of length. The relation between the *gram* and the *centimeter* is easy to grasp; and the relation between the *kilogram* and the *liter* is also easy to grasp. Moreover, the relations are so simple that the pupil can move back and forth in his thinking from one kind of measure to another. Once he has visualized a centimeter, and then, ten centimeters, in length, he can visualize a kilogram, or a liter, without difficulty, and shift his attention from one to the other with ease. He is aided in doing this, not merely by what he remembers and understands about relations, but also by the fact that he can easily carry the relations in his mind. They all are stated in the decimal system. Since the idea of ten has become so familiar, and since the metric system is a decimal system, the pupil does not need to hesitate in transferring his attention from one measure to another, or from one unit within a measure to another, until he can use pencil and paper computations to aid him. The presence of the idea of ten in the system invites an easy movement of thought from one measure, or any part of one measure, to another or to any part of another.

3 Importance of the Method of Teaching

Much depends, of course, upon how the metric system is taught. If the teacher is interested only in values that are derived from use, he will find little or value in making impressions. Indeed, the metric system is so lacking in usefulness to the pupil of the elementary school as to make the teaching of what little use it may ever have an extraordinarily difficult task. The teacher who tries to teach usefulness will usually find himself so engrossed in the task of trying to discover uses to exhibit and illustrate as to forget entirely the valuable relations between measures necessary that the system can make clear. Thus, the teacher may even actually go back from the pupil's view the valuable relations that otherwise are so obvious and so clearly evident.

On the other hand, if the teacher is interested in helping pupils to develop an understanding of measurement, of what measurement involves, of the precise thinking that originates in measurement, and of how the work of the world has moved in the direction of precision through the development of measurement, he will use every means possible to call attention to the relations between measures and to illustrate such relations. He will refer to ancient measures whose use has long since been abandoned, in order to suggest the need of relations; he will indicate the slow development of measures from diversity to uniformity, in order to impress the importance of growing relations upon the development of precision; and he will introduce his pupils to a system of measurement in which the relations are so obvious and so clearly expressed as to make the transition of thought from one measure to another a matter of comparative ease, in order to provide needed practice in thinking the relations between measures. The teacher who is interested in transmitting to the pupil an idea of how the race has progressed toward precision in thinking through the development of standard measures that make possible the translation of the

1. Measurement of Lengths and Linear Measures

The modernized figures with a review of general measures as an introduction lead to specific measures.

An Illustrative Lesson on Measuring Lengths

You have already learned a good deal about how to measure. You know, for example (although you may not have thought about it just like this) that the way to measure the amount of a certain thing is to take the right kind of a measure and apply it to the thing to be measured. Then, the way to measure liquids is to take a quart, pint, or gallon measure and apply it (count the times the liquid fills the quart, pint, or gallon measure). The way to measure grains, vegetables, etc., is to take the quart, peck, or bushel measure and apply it (count the times the material fills the quart, peck, or bushel measure). The way to measure lengths is to take the foot, yard, or rod measure and apply it (count the times the length being measured, not exactly like, but outside over the foot, yard, or rod measure).

In order to measure lengths we must first have the right kind of a measure — the inch, foot, yard, or rod. Next, we must apply the measure to the length, and count the times the measure has to be applied to cover the length. Let us now.

In order to measure her height, Jack stood up against the wall and marked on the wall level with the top of his head. He took a foot rule, applied it to the length from the floor to the mark on the wall, and counted the feet as he covered the distance: one, two, three, four, and a half — $4\frac{1}{2}$ feet, or he counted the inches as he covered the distance: 12, 24, 36, 48, and 6 are 54 inches.

In order to measure the width of the vegetable garden, Bob took the yardstick, and applied it to the distance from corner stake to corner stake, and counted the yards, or the feet, as he covered the distance: 1, 2, 3, 4, 5, 6, 7, 8, 9 yards — or 3, 6, 9, 12, 15, 18, 21, 24, 27 feet.

That is the way people always have measured lengths or distances: Take a measure and apply it. Of course, in olden days the measures were not so exact as the ones we use now; yet we have got our modern measures and the method of applying them from the measures and method of former times.

measures, with us we now verify that that is what they had in the mind of the child, and thereby give us a measure of length before they could measure length directly. When they had decided upon a measure, the next thing to do was apply it. But we remember these two fundamental things that we had to do in order to measure lengths, or anything else, directly:

1. Decide upon a measure.
2. Apply the measure.

Measuring Surfaces

Bob and Betty were interested in the size of the whole vegetable garden, the whole playground, and all the space in their rooms, and just the dimensions they were long and wide. How big is Betty's room? Of course Betty knew, when she had measured the room, just as we would know it, that the room was 12 feet long and 8 feet wide. And, of course, in measuring, she just measured along two sides of the floor. What she wanted to know was how big is the whole surface of the floor, and merely how big it is along two sides.

In measuring the garden, Bob measured first along one edge of it, then along another edge. But Bob wanted to know how big the whole garden is, not merely how big it is along its edges.

The garden, the playground, and the floor in the two children's rooms are surfaces, not lengths. A surface has length just as a piece of thread has length, but the surface of a garden is quite different from the length of a thread.

We must learn how to measure surfaces. In order to tell how big a surface is, we must do two things:

1. Decide upon a measure of surfaces.
2. Apply the measure.

Let us now do each of these in turn.

2 Method of Procedure in Square Measure

When the pupil has in mind the two steps of the procedure he is to undertake -- namely, to decide upon a measure of surfaces, and to apply the measure -- he may proceed to take the two steps in turn. The first task is to become acquainted with the *square foot*, the convenient unit of square

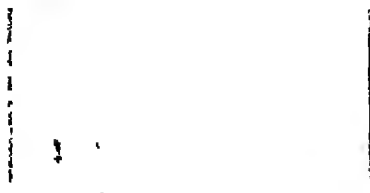
measure. To do this the pupil must be led to study the openings between lines, and the particular angle opening, which is called the right angle point. He must give his attention to 'square-cornered' surfaces, including the particular kind, called 'squares'. He must study the characteristics of the square, and finally, he must give repeated attention to the square foot, making for his own use of cardboard a square that approximates as closely as possible the square foot. Only by taking all these steps in succession can the pupil become familiar with the peculiar properties of the square and with the exact value of the square foot.

Now, when he has selected his measure of surface, he may proceed to apply it in the actual measuring of surfaces. That of the actual measuring of surface, he may be led to discover the rule of the rectangle, which provides an indirect, but very convenient, means of measuring surfaces.

A Lesson on Applying the Square Foot

1. Draw on the blackboard a rectangle exactly 2 feet high and 4 feet long. (Perhaps your teacher will have it drawn for you.)

How large is this rectangle? The question asks: How many square feet are there in the rectangle? Apply your measure, the square foot, and find out.



Lay your square foot in one of the corners, say at 1, so that it just covers the corner, and make marks on the rectangle to show how much space is covered. That is 1 square foot. Measure another square foot space, and mark it off. That will be another square foot. Continue in this way to measure the space on the rectangle, counting the square feet, until all the space is measured. (Do not miss any of the space in the rectangle, and do not measure any part of it twice.)

When you have finished measuring the rectangle that is 2 feet high and 4 feet long, you will have squares, each a square foot in size, marked all over it like this:

How large is that rectangle that is on the board? That is, how many square feet in size is it?

2. Draw a square (rectangle) on the board that is 3 feet high and 3 feet long. Measure it, and write down how large it is.

3. Draw a rectangle on the floor 3 feet wide and 4 feet long. Measure it, and make a record of its size (write down how many square feet are in it).

4. Draw another rectangle on the floor 4 feet wide and 5 feet long. Measure it, and record its size.

5. Draw a square with sides each 5 feet long. Measure it, and record its size.

6. Draw a rectangle 5 feet wide and 6 feet long. Measure it, and record its size.

7. Draw a rectangle 9 feet wide and 12 feet long. Measure it, and record its size.

Using your square foot as a measure, measure the size of other rectangles that your teacher may have drawn for you. As you measure each rectangle, you may make a table like the one shown below to use in recording the size.

Perhaps you can already tell an easier way of finding the size of a rectangle than the way of actually measuring it with your square foot. If you cannot, you should study the table. Let us study it together.

1. When you measured the first rectangle on the board, which was 2 feet high and 4 feet long, you found that the size was 8 square feet. You learned by actual measuring that 8 square feet is the exact size. Look at the two lengths, 2 feet and 4 feet. Can you tell what one should do with the numbers 2 and 4 to get 8? If you can tell, you do not need actually to measure the surface with your square foot.

Number	Lengths of Sides		Area of Rectangle Number of Square Feet in It
	Feet Wide	Feet Long	
1			
2	3 feet	3 feet	9 square feet
3	3 feet	4 feet	12 square feet
4	4 feet	5 feet	20 square feet
5	5 feet	5 feet	25 square feet
6	5 feet	6 feet	30 square feet
7	6 feet	12 feet	108 square feet
8			
9			
10			

Study the other figures, and compare the area of the sides with the area of the surface, as given in the table.

1. What should one do with 3 and 3 to give 9?
2. What should one do with 3 and 4 to give 12?
3. What should one do with 4 and 5 to give 20?
4. What should one do with 5 and 5 to give 25?
5. What should one do with 5 and 6 to give 30?
6. What should one do with 6 and 12 to give 108?

Perhaps you can now make your rule for finding the area of the surface of a rectangle, without actually having to use the square foot to measure it. The rule will be something like this:

To find the area of a rectangle in square feet, multiply the number of feet long by the number of feet wide.

Likewise:

To find the area of a rectangle in square yards, multiply the number of yards long by the number of yards wide.

To find the area of a rectangle in square inches, multiply the number of inches long by the number of inches wide.

To find the area of a rectangle in square rods, multiply the number of rods long by the number of rods wide.

3. Studying Other Measures of Surface

When the pupil has developed the square foot as a measuring instrument, has used it in the measuring of surfaces, and, from the four stages of perimeters, has developed the rule of the rectangle, he should continue his studies of surface. The following materials suggest three methods: (1) comparing surfaces with perimeters, (2) determining the relations between the various derivatives and multiples of the unit of square measure and (3) studying through familiar comparisons the sizes of typical surfaces.

Training the Distinction between Distance Around and Size

You have learned how to find the size of any surface that is shaped like a rectangle. The size is measured in, and stated in, square feet, square yards, etc. The size of a surface is not the same as the distance around it, although some children confuse the two. Let us make the difference.

1. Mr. Henderson was planning a little chicken yard at the end of his lot. He staked it off (stake little stakes at the four corners) and measured between the corners to see just how long it was. The chicken yard measured 20 feet wide and 20 feet long. He now built a fence around the chicken yard. How long is the chicken yard? How long is the fence that goes around it?

Let us draw a diagram of the chicken yard. It will be like this:



How long is the chicken yard; that is, how many square feet in it? $20 \times 20 = 400$ square feet. How long is the fence that goes around it; that is, how many feet around it?

The distance from A to B is 20 feet.

The distance from B to C is 20 feet.

The distance from 1 to 2 is 20 feet

The distance from 1 to 3 is 20 feet

The distance all the way around is 100 feet

Then we get our answer

The area of the square yard is 400 square feet

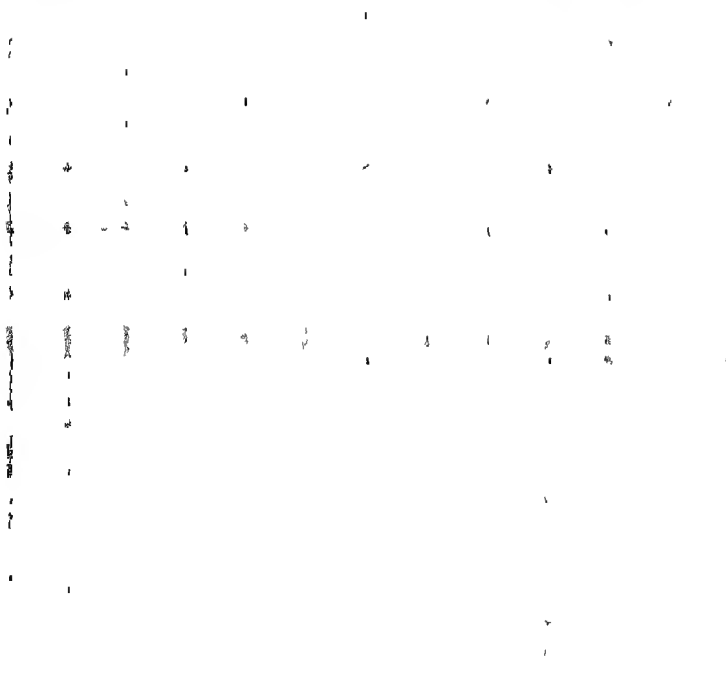
The distance around the square yard is 100 feet

Now whether you can tell another way of finding the distance around.

Teaching Square Inches, Square Feet, Square Yards

1. How many square inches are there in a square foot?

Draw on a piece of paper as we mentioned a square foot. Use the square foot you used as a guide again to measure the area of square inches before you learned the rule.



Using your rule, mark the edges off exactly in inches, as shown in the diagram, and connect with straight lines the opposite points that you have marked. When you get all the lines drawn you will have the square foot marked off into little squares. How many in

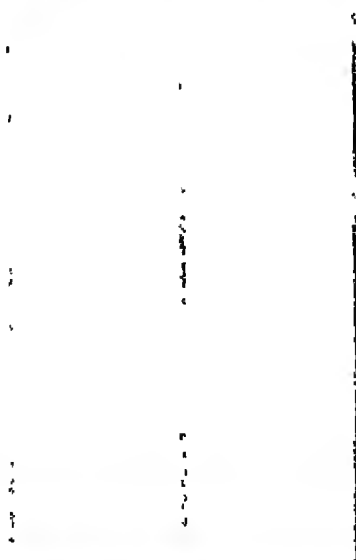
each little square? The sides of each little square are exactly 1 inch long, or each is a square inch in size.

Count the square inches in the square foot. This is the way to do it. Count the square inches along the top of the bottom. How many square inches in 1 row? Next, count the number of rows. Then multiply the number of square inches in one of the rows by the number of rows. $12 \times 12 = 144$ square inches.

A shorter way to find the number of square inches in a square foot is to measure the length of the square foot in inches, and the width in inches. You will find that the square foot (perhaps you already know it) is exactly 12 inches long and 12 inches wide. How many square inches are there in a surface that is 12 inches long and 12 inches wide? $12 \times 12 = 144$.

2. How many square feet are there in a square yard? A square yard, we know, is 3 feet long and 3 feet wide. How many square feet are there in a surface that is 3 feet long and 3 feet wide? $3 \times 3 = 9$ square feet.

Draw a square yard on the blackboard. Perhaps your teacher will have it drawn for you. Mark the square yard off into square feet, as shown in the diagram.



3. How many square yards are there in a square rod? You can think how to find the answer, like this: A square rod measures $5\frac{1}{2}$ yards long and $5\frac{1}{2}$ yards wide. There are then $5\frac{1}{2} \times 5\frac{1}{2}$ square yards in a square rod, or $30\frac{1}{4}$ square yards.

Compare the tables given below

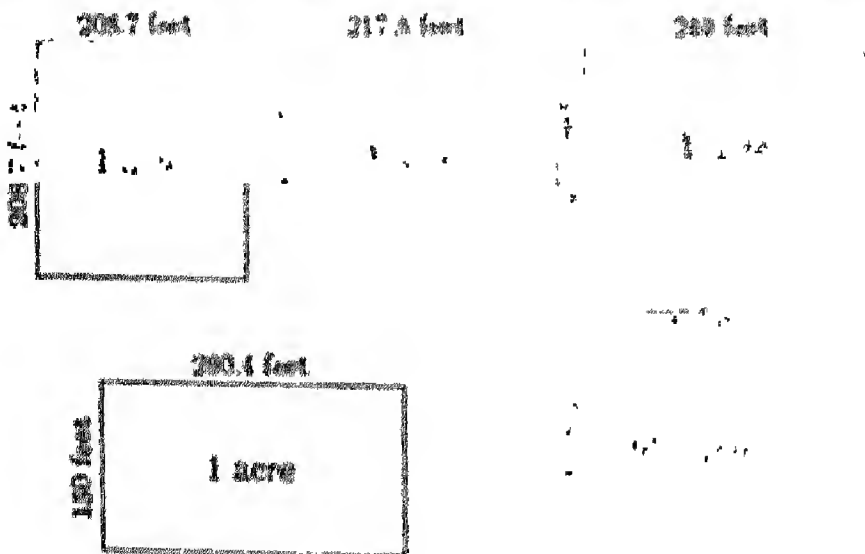
Table of Length	Table of Square Measures
12 inches = 1 foot	144 square inches = 1 square foot
3 feet = 1 yard	9 square feet = 1 square yard
5 1/2 yards = 1 rod	30 1/4 square yards = 1 square acre
320 rods = 1 mile	160 square rods = 1 acre
	640 acres = 1 square mile

A Lesson to Show How Large an Acre Is

Acres were measured carefully and square rods, or a piece of square land, later, were used as an indirect measure of land measuring the amount of land a man could plow in a day with a team of oxen. Still later, were used for measuring a strip of land 4 poles wide and 40 poles long (bush up distances of acres in the Surveyors' Office). Finally the acre has come to mean a piece of land exactly 160 square rods in size.

We use the acre today as a measure of ground of land that is larger than the ordinary city lot. The acre is 160 rods, or 40 rods, or measured in acres. The square of a man covering a piece of 40 rods or 160 rods, or 160 acres, or 160 acres.

An acre is the size of a square the sides of which are slightly more than 200 feet in length. It is the size of a rectangle that is 200 feet wide and 217 1/2 feet long, or of one that is 176 feet wide



and about 300 feet long, or of one that is 300 feet wide and 200 feet long. A square with sides 200 feet in length is slightly more than 1/2 of an acre.

There are 43,560 square feet in 1 acre.

Measure the front of your school lot or road. Divide 43,560 by the number of feet the lot measures in front. The number you get is the number of feet to measure back on your school lot to give a piece of ground the size of 1 acre.



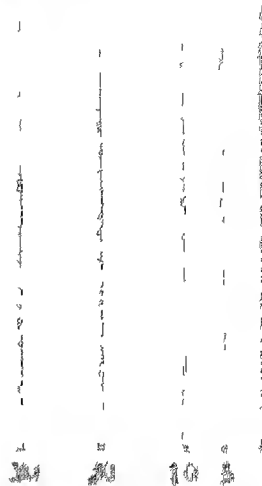
Let us suppose that the front of the school lot measures 300 feet. 43,560 divided by 300 gives 145.2. Measure back 145.2 feet, and the square above that is 300 feet long and 145.2 feet wide is 1 acre in size.

The next time you go to a football game, notice the distance between the side lines. This distance is 160 feet. Then notice the distance between the two 5-yard lines. This distance is 90 yards, or 270 feet. $270 \times 160 = 43,200$ square feet, which is almost, but not quite, 1 acre.

The size of the playing part of the football field -- between the side lines and between the two goal lines -- is slightly more than 1/2 acre.

Whenever you want to picture in your mind the size of an acre, just picture the football field between the side lines and from 5-yard line to 5-yard line. Or, just picture a baseball diamond, and

10 20 30 40 50 60 70 80 90 100



It is not clear from the text what the graph represents, but it appears to be a series of measurements or data points plotted against a vertical axis.

THEORY OF THE EARTH AND ITS HISTORY

1. A field of 100 acres is divided into 10 equal parts. If one part is 10 acres, how many parts are there in the field?

2. A certain number of 100 acres is divided into 10 equal parts. If one part is 10 acres, how many parts are there in the field?

3. A field of 100 acres is divided into 10 equal parts. If one part is 10 acres, how many parts are there in the field?

4. A field of 100 acres is divided into 10 equal parts. If one part is 10 acres, how many parts are there in the field?

There are 10 equal parts in the field.

5. A field of 100 acres is divided into 10 equal parts. If one part is 10 acres, how many parts are there in the field?

THEORY OF THE EARTH AND ITS HISTORY

Sometimes a person is asked to find a certain number in a square field, square garden, or in a large number of square rods and acres. In a square field, square garden, square rods, square garden,

squares with an object's surface: he wishes to have the size stated, he uses the corresponding measures of length (rod, yard, foot, or inch); he measures the length of the sides.

Suppose you wish to find how many square yards of floor covering it would take to cover the floor of the library reading room at school. You would measure the lengths of the sides in yards. Suppose you wish to know the size of your living room at home in square feet. You would measure the lengths of the sides in feet. Suppose a farmer wishes to know how many square rods, or how many acres, there are in one of his fields. He would measure the lengths of the sides in rods. Suppose you wish to find the size of a park in square miles. You would measure the lengths of the sides in miles.

1. How large is your schoolroom floor in square feet? In order to find the answer to this question, what must you do? Find the number of square feet.

2. How large is your schoolroom floor in square yards? In order to find the answer to this question, what must you do? Find the number of square yards.

3. Can you suggest another way of finding the number of square yards in your schoolroom floor without actually having to measure the lengths of the sides a second time?

XI DEVELOPING A METHOD OF ATTACK

The material just presented illustrates what appears to be a very roundabout manner of developing the rule of the rectangle when it is remembered that the pupils can memorize the rule in a five-minute exercise. If it were merely the rule of the rectangle and its use in finding answers in which we are concerned, we could readily choose the memory exercise in preference to the long-drawn-out series of development lessons here illustrated. Our concern, however, is not in getting the pupils to acquire a rule-of-thumb procedure, but in helping them to understand square measure and to develop a method of attack upon the various problems of measuring surfaces. The method of attack upon the rectangle may not be so important if rectangles were the only kind of surfaces

to be measured. There are other kinds of surfaces, however, to which the pupils must give their attention. When the pupils have developed a general method of attack, they may proceed to the study of surfaces other than the rectangle.

The material shortly to be presented illustrates how the pupil may proceed to the measurement of the parallelogram that is not a rectangle, the triangle, and the circle. It is supposed that the properties of each of these surfaces have already been studied. The problem confronting the pupils is how to apply the unit of square measure first to one, then another, of the surfaces indicated. When they study the first kind of surface, they meet the difficulty of applying the unit of square measure to it. They are confronted with the problem of changing the figure so that the unit can be applied. In each succeeding surface studied a similar difficulty arises, and the same problem is met. Whether the pupils solve the problem in each case or are assisted in solving the problem is not material. The matter of paramount importance is whether the pupils understand the problem that needs solution. Since the problem was first presented as a method of procedure in connection with the rectangle, and since the problem arises again and again in each of the surfaces to be studied, the problem takes an more and more definite shape in the minds of the pupils. The solution will be understood, however it may be secured, if the problem is clear. The following material will illustrate the procedure in bringing the problem again and again to the attention of the pupils.

Teaching Pupils to Apply the Unit of Square Measure

As you already know, one can find the size of a surface that has square corners, like the square or the rectangle, either by applying the unit of square measure — square inch, square foot, square yard, etc. — or by multiplying the length by the width. If the figure has square corners, one can apply the square foot (and the other square measures), because the square foot will fit

exactly into the corners. But suppose the surface is one that does not have square corners, like the parallelogram, the triangle, or the circle. In such a case, one forms a problem. Let us see what the problem is.

Draw the following figures on the blackboard with the dimensions shown in the figures below.



Now take a square foot - a piece of cardboard that is exactly 1 square foot in size - and try to apply it to each of the figures. How many square feet there are in each of them.

One may try and try to apply the square foot to the figures, and he will find that the square foot cannot be applied exactly and entirely to any one of the figures, because in each case the square foot will not fit exactly and entirely within the figure. In each case, the square foot sticks out at some point or points. What can one do? How can he ever find the number of square feet in each figure as the parallelogram, the triangle, and the circle?

There is one way, and only one way, and that is, if it is at all possible, to change the figure in which the square foot does not fit into a figure into which the square foot will fit.

Into what kind of figure will the square foot fit? The square foot (square inch, or square yard, etc.) will fit into any figure that has square corners, such as the rectangle or the square.

Now, if one wishes to measure the size of the parallelogram, the triangle, or the circle, he can either give up the task in defeat, or he can try to find some way to change the figure into one with square corners, like the rectangle. The problem one faces in measuring such figures as the parallelogram, the triangle, or the circle, is first to change the figure to a rectangle with square corners; that is, into a figure to which the unit of square measure can be applied.

Let us try to make this proof more exact. Let us draw a line through the point P parallel to the base BC of the triangle ABC . This line will intersect the sides AB and AC at the points D and E respectively.

What is the parallelogram?

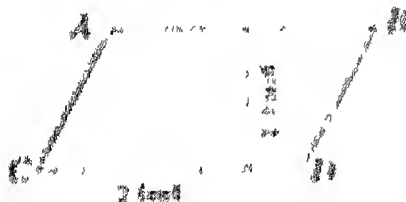
Draw a line through the point P parallel to the base BC of the triangle ABC . This line will intersect the sides AB and AC at the points D and E respectively. The figure $ADPE$ is a parallelogram.

Measuring the Parallelogram

When you measure the perimeter of the parallelogram $ADPE$ by applying the unit of measure, you find that the area of the parallelogram is equal to the area of the rectangle $ADPE$. This is because the parallelogram can be changed into a rectangle by cutting it along the line DE and moving the part $ADPE$ to the right.

What can you do to change the parallelogram into a rectangle? Let us see.

Draw a parallelogram like $ADPE$ on a piece of rectangular paper. Cut it out along the lines AD , DE , and EP .



Draw a line from A to the point P on the side BC of the triangle ABC . This line will intersect the side BC at the point P . The figure $ADPE$ is a parallelogram. The figure $ADPE$ is a rectangle. The area of the parallelogram is equal to the area of the rectangle.

When you move the triangle $ADPE$ to the right, what kind of figure do you have? How long is it? How wide is it?

Since you know how to find the area of a rectangle by multiplying the length by the width, you can find the area of the parallelogram into which you have changed the parallelogram, and when you find the area of the rectangle, you will know the area of the parallelogram.

parallelogram, with which you started. Whenever you change a parallelogram to a rectangle, the area is the same. The rectangle that is as long as the base of the parallelogram and as wide as the altitude of the parallelogram. Hence the area of the rectangle is

$$\text{Area} = \text{length} \times \text{width},$$

the area of the parallelogram with which you started is

$$\text{Area} = \text{base} \times \text{altitude}$$

Measuring the Triangle

When you attempt to measure the area of the triangle by applying the unit of square measure, you find that the unit of square measure will not apply exactly. You find that you must first change the triangle to a figure, like the rectangle, to which the unit of square measure can be applied.

What can you do to change the triangle to a rectangle? Let us see.

Draw a triangle like ABC on cardboard or paper.



Cut it out along the lines AC , AB and BC .

Draw another triangle that is exactly the same size, and cut it out.

Put the two triangles together as shown here:



What kind of figure do you now have? How long is it? How wide is it? How do you find its size?

How does the area of the triangle with which you started compare with the area of the rectangle you now have? If you know,

we can find, the area of the parallelogram is $2 \times$ the area of the triangle with which you started. Now, if you know the base and height of a triangle, you can find the area of the parallelogram.

Since the area of the parallelogram is

$$\text{Area} = \text{base} \times \text{height}$$

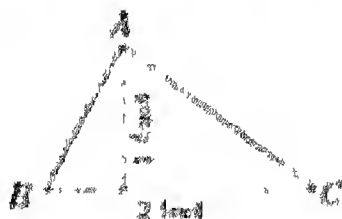
and since the height of the parallelogram is the same as the height of the triangle and the width of the parallelogram is the same as the width of the triangle, the area of the triangle is

$$\text{Area} = \frac{\text{base} \times \text{height}}{2}$$

or

$$\text{Area} = \frac{1}{2} \times \text{the base} \times \text{the altitude}$$

Suppose now we know the base of a triangle and its height and have an equilateral triangle like the triangle you have just been studying. How can you find the area of a triangle like this one? How to find the area of a triangle like this one.



Draw a triangle like the one above, and cut it out. Then another triangle exactly the same size and cut it out. Place the two triangles together, and you will have a figure that looks like a parallelogram. The two figures will look like this:



that is, like a parallelogram that has a base 2 feet long and an altitude 1 foot long.

Since you already know that the area of a parallelogram is

$$\text{Area} = \text{base} \times \text{altitude}$$

and since the size of the triangle with which you started is exactly

we find the area of the parallelogram, that gives twice the area of the triangle is

$$\text{Area} = \frac{\text{base} \times \text{altitude}}{2}$$

or

$$\text{Area} = \frac{1}{2} \text{ of the base times the altitude}$$

Measuring the Circle

We have now succeeded in measuring the area of the circle by applying the area of square measure. We think that the area of square measure will not agree exactly. We think that we must first change the circle to a figure, like the rectangle, in which the area of square measure can be applied.

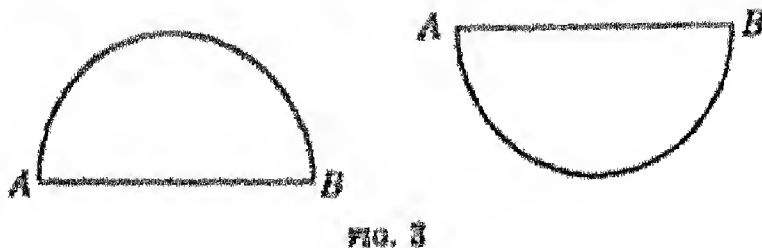
In order to change a circle to a rectangle, we must be able to use the measurement of it. Let us see how well you are able to use your imagination.

Suppose we try to make the circle into a rectangle.

First, let us divide the circle into halves by drawing a diameter through the center, thus:



Next, let us set these two halves off by themselves, thus:



Notice that the distance between the two ends of the half-circle is exactly half of the circumference. We now proceed to the next step.

Now, let us draw lines from the center of each half-circle, dividing each into smaller parts, as we



FIG. 4

Let us imagine that many, many circles have been drawn on each half-circle, dividing each into many, many parts.

Now, let us imagine that we can take half of the radius of the half-circumference at A and at B in the first half-circle, and push it out into a straight line, thus



FIG. 5

The straight line AB is half of the circumference straightened out. Hanging down from the straightened out half-circumference AB are one-fourth parts of the circle, each of which is equal in the radius in length.

Let us do the same straightening out with the second half-circle, and we shall have a figure like this

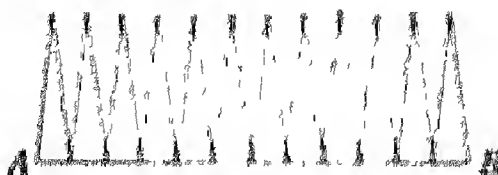


FIG. 6

which is exactly the same as the first one. Now, let us imagine that we fit the two straightened out half-circles together. As we imagine pulling them together, we can see how the "one-fourth" of

that area is $\frac{1}{2}$ of the area under the curve. We now have a figure like Fig. 7.

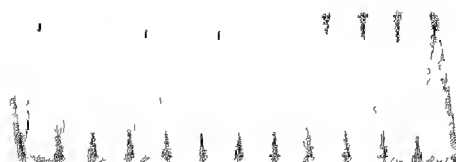


FIG. 7

which is now half of the circumference being used as wide as the radius.

How large is a circle? If you have used your imagination well, you can tell that the area of a circle is $\frac{1}{2}$ of the circumference times the radius.

XII THE MEASURE OF MEASUREMENT

Our discussion in this point carry the suggestion that measures and weights are not merely concrete instruments that the individual may pick up for certain practical uses and then lay aside and forget but are in reality the concrete representations of a method of thinking that man has slowly and gradually developed and that he has used to transform the world in which he lives. The yardstick is such a common thing as to be found in every household; anyone can possess it and learn to put it to certain practical uses. But the yardstick is more than a piece of furniture; it carries within it the story of man's rise from savagery to civilization; it is a kind of milepost along the pathway of human progress.

What measurement has come to mean in the story of human progress is indicated in the following quoted paragraph from the concluding statements of Judd's chapter on the psychology of precision.²² The statements are quoted at length because they assemble in compact form a number of points that have been merely implied in our discussions,

²² From C. H. Judd *The Psychology of Social Institutions*. (By permission of The Macmillan Company, publishers, New York, 1926.)

the first generalization is that in the early stages of an evolutionary series, the purposes which prompt men to give attention to the system gradually change. In the first stages of the evolution of weights and measures the purposes were wholly practical. In later stages the desire for uniformity

became more prominent. The second generalization is that in the early stages of an evolutionary series, the purposes which prompt men to give attention to the system gradually change. In the first stages of the evolution of weights and measures the purposes were wholly practical. In later stages the desire for uniformity

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throughout it is plain that the authority of organized society was needed in its complete and systematic development. Because some kind of control of the kind which they thought necessary in order to prevent a general relapse in a still later stage the more advanced stages of social organization responsibility features all standards to themselves as rather than thought may be made general in the full sense of the word. The long and continuous was the only source for the growing complexity of weights and measures, even and very well with enough approximation. But when they came to think about the dimensions of the earth and the vast depths of light they saw that the measurement of different measures would be made comparable, and they saw also that all the world would be made sufficiently precise to make rational judgments possible. Measurement becomes in these last stages the highest form of controlled thinking. Systems of measurement take on a kind of method and a kind of unity which could authority, unsupported by scientific authority, could never give them.

In the evolution of weights and measures we have another striking illustration of the fact that man has introduced into his environment an institution which essentially changes his relations to nature. He has created by the exercise of his power an instrument for the guidance of his behavior which insures economy of material and effectiveness of conduct in the highest degree. At the same time he has set up a guide for individuals and a means of compelling the individual to conform to the practices of society, which are among the most potent factors of civilization.

It may not be amiss to emphasize even more the fact that there is absolutely no instinct in the animal world or in human nature which responds to weights and measures. It is utterly futile to attempt to bring these creations into the class of biological facts which are dealt with in the ordinary applications of the principle of biological variations and natural selection. Weights and measures constitute one of the institutions of civilization. The history which has been sketched in the foregoing pages shows what a vast amount of energy has been bestowed on the erection of this institution. The history of men's thinking which parallels the facts

learn that seven times six is forty-two is secondary in comparison with the more important matter of right. In short, then, the content of the course will be considerably changed. There will not be a great deal for the pupil to learn, because the teacher will have difficulty in presenting the possession of a good many lessons merely on the grounds of usefulness. The content will in many cases be reduced to those commonly used measures and weights with which the pupil, when he chooses to come into contact with these uses, can facilitate himself just about as well without the benefit of school instruction. In such case, the study of measures and weights might just as well be excluded from the curriculum in arithmetic. It is worth considering whether the school is ever justified in including a topic in its curriculum unless it is prepared to carry the development of the topic to greater lengths than is possible in the daily affairs of life.

In the other hand, the teacher may be interested in teaching the uses of common measures and weights and the computations their uses require only as a prelude, or an introduction, to the learning of measurement as a human invention that has helped to shape the course of human progress. In such case, uses of measures and useful computations will not be neglected; indeed, they will receive an emphasis that a narrowly conceived course cannot give. The pupil will not merely acquire some devices that are useful to other people; he will also develop the knowledge and skill that will make them useful to him. Moreover, whether the pupil may forget the devices before he has a chance to use them will cease to be a source of worry. His gaining of meanings will supply the pattern into which the useful devices may be set and that may enable the problem of memory to solve itself. What is more to the point, however, is the fact that the topic on measures and weights may become for the pupil a chapter on the progress of civilization. The chapter admittedly will not be a completed one in the case of the elementary pupil. It will, however, present some things

that he can now and again make good it in a number of instances of others that may be expected in his future organization. The extent of his work may not be as great as he intended it is certainly well not exceed the time he himself has been able to give.

CHAPTER XX

HELPING THE RETARDED PUPIL.

ARITHMETIC

1. Many pupils fail in arithmetic because of arithmetic and computational

2. Pupils in other parts of the course indicate that it is realized that a well-developed ability to deal with failure should be made

3. Illustrations of various failures are given from the report of Howard and Jones

4. The failure of a pupil is as much a general condition as it is a specific one. It frequently is the result of learning, understanding, and of the work, and of the work of the work

5. A pupil will never know the pupil's work, it may fail to know the reason for which he has failed. It is well to point out to the pupil that understanding does not always accompany accuracy and speed of performance but frequently appears independently of accuracy and speed

6. The results of tests may be very misleading. It is as important for the teacher to discover from the pupil's performance as to discover what the results are

7. Remedial instruction is frequently an immediate necessity, it generally does not strike at the roots of failure.

8. Remedial instruction ought not to be mistaken for curative instruction. To be curative, instruction must strike back at evident difficulties in their causes

9. Frequently the cause of failure is removed when the review of poorly understood processes is so devised as to provide a new, and an enlarged view. New views and enlarged views serve the needs of the pupils who are succeeding as well as of those who are failing

10. Lesson materials are offered to illustrate what is meant by new and enlarged views and how they may be provided.

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More people fail in arithmetic than in any other elementary subject. Pupils with sufficient intelligence for the work, failing in the use of their number language and the operations that are required between numbers and their practical application to everyday life, show a surprising lack of consistency of knowledge. Pupils who have taken the first steps in arithmetic, the multiplication, progression, and by a mastery of counting, progress in the construction of the most important difficulties in their progress in their progress in the succeeding stages. In spite of the fact that the understanding and use of number seems so easy to the child and so simple when he has succeeded, the learning of arithmetic seems an almost impossible task to others.

There appears to be little reliable ground in this matter of learning arithmetic. Pupils either learn and never let it go or they do not. They either succeed or fail. In either case, some may find pupils distributed all along the road from failure to mastery. A few at the point of failure never at the point of approaching "average" attainment, a large number at the middle point of "average" attainment and on many more may be called "superior," and almost the entire number of successful attainment as of failure. In arithmetic, however, pupils seem to be grouped rather on one side or the other of a dividing line between success and failure. For the most part, there are partial failures and extreme failures as well as successes of varying degrees of success, but, in the main, the pupils either understand division or percentage, for example, or they fail to understand.

Failures in arithmetic are cumulative. Each succeeding grade witnesses a larger number of failures than the grade preceding. The partial failure of a pupil in the primary grades usually turns out to be a complete failure in the intermediate grades. In other subjects, the partial failure at a given point may be turned into success at a later point. If,

For instance, a pupil does great work in his study of the properties of addition, but as soon as he is asked to multiply he soon does much better work in his study of simple fractions. In arithmetic, however, the various topics are so intimately related that disorientation from the main road at any point results in extraordinary difficulty in getting back on the road at a later point. The pupils depend upon continuous travel over the main road that the pupil must usually be taken back to the point of original disorientation before he can be set moving again in the right direction.

II. ILLUSTRATIONS OF FAILURE

Teachers are aware of the presence of retarded pupils in their classes in arithmetic. They often are at a loss about the way to deal with them. The following extreme cases of failure have been reported by Huswell and John,¹ not as illustrations of typical performance in elementary-school classes, but as illustrations of the extremity to which failure is permitted to proceed without any well-planned attempt on the part of the school to deal with it. We quote at length from Huswell and John's report:

Tina, a pupil in the third grade, was rated by her teacher as poor in arithmetic. Her work in subtraction illustrates some of her difficulties. In the case of the example, 53 minus 4, she first tried to get the answer by counting. Beginning at 51, she counted, "51, 52, 53, 54," but she was not able to decide what the answer should be. She said that she wished she had a little paper. When a piece of paper was given to her, she made fifty-eight marks on it and then crossed out three. She counted the remaining marks and gave her answer as 55. When she was asked why she had crossed out three marks instead of four, she replied that she had forgotten one. After crossing out another mark, she counted the marks a second time, again giving the answer 55. When her work was

¹ U. T. Huswell and Lenore John. *Diagnostic Studies in Arithmetic*. (Department of Education, The University of Chicago, Chicago, 1926)

checked, it was found that she had made some small errors and had corrected them in the first time. This indicates that the method of working makes much difference, even in the method of making simple sums. The first example was as follows: When the examples given for the first examination were made smaller numbers, she found no trouble in the first. In the case of the example, it was 9 and 10 and the answer by imagining an extra hand and using two fingers on the third hand twice to represent the number 10. She then counted off all the fingers on one hand and four fingers on the other to equal the number 10 which was in her mind. She gave her answer as 1, only one finger remaining on each. She seemed surprised, thinking that this was wrong, and quickly asked for a piece of paper, on which she wrote 10, 10, 10, as in the case of the example given. She then said, "I think Tula could not be expected to show any great efficiency in arithmetic as long as she used such methods as these."

Jane, who also had considerable difficulty with calculation, was in the fifth grade. In calculating, she generally counted backward to get her answers, as a result, she made frequent errors. A typical illustration of her work appears in the example, 15-10 minus 10. A statement, report of the work in which she obtained her answer is as follows: "I began 15, 15, 15, 15, 15, 14, 13, 12, 11, put down 1 and carry 1, 1 from 11 leaves 12, 12, 11, 10, 9, 8, 7, 6, and 1 in 7, 6 from 14 leaves 15, 14, 13, 12, 11, 10, put down 0 and carry 1, 1 from 8 is 0 and 1 is 1." Jane was far from perfect in manipulating even the clumsy technique which she had adopted, as evidenced by the errors made in this example.

Kurt was a bright boy in the third grade who had received his previous instruction at home and who returned school for the first time in this grade. He experienced many difficulties with both addition and subtraction. Some of his difficulties were of a peculiar type, especially those relating to his reading of numbers. In the example, 55 minus 4, he read the 55 as 85, then counted back to 81, and wrote the answer 18. He was very frequently confused by 6's and 9's, not being sure which was 6 and which was 9. In the example, 79 minus 1, he said, "87," both inverting and reversing numbers. He

multiplied back 45 by 27, obtained 1215 and made the answer 76. In the second example, 48 divided by 2, he wrote the 24 as 48 and then began to subtract much as follows: "48, 76, 24" He then continued 48, 76, 24, 48, 24, and continued as follows: 48, 76, 24, 48, 24. He continued the 48 and wrote the answer as 24. In the case of the example, 48 plus 7, he said, 48 plus 100, 102, 104, 106, 108 and wrote 108 as the answer. In subtracting 48 from 108, he said, "48 and 6 in 48, 108, 48, 72, 48, 24. And I don't know which remainder there is (pointing back to the 48). Is it 48 or 24?" He then continued, "48, 24, 48, 24, 48, 24." In repeating the third answer, he repeated the digits and wrote 48. The methods and processes which he used were such as, namely: He wrote showed an interesting picture of mental processes and resources before. He showed that a child's mind is not a simple process of calculation but a complex one. He showed that the child's mind is not a simple process of calculation but a complex one.

The same was a fifth-grade boy who was called by his teacher as given an arithmetic. In the case of division he showed a number of strange methods. He repeatedly added the remainder from one division to the next stage of the process in the next stage of the quotient. For example, in dividing 17385 by 2, he said, "2 into 17 goes 8 times and 1 over, 2 into 3 goes once and 1 (the previous remainder) is 5, 2 into 5 goes 2 and 1 over, 2 into 3 goes once and 1 (the second remainder) is 2 and 2 over. The answer was 8692½. He used this method in all examples in short division in which there were remainders. He frequently continued in order to get the quotient. In the case of the example, 48 divided by 2, he followed this complicated process: "2 into 48, 12, 12 more 2's make 48, go back 2 to 48, make 24." He continually used the short-division method for examples which should have been worked by long division. In the case of the example, 16384 divided by 34, his answer was $425\frac{24}{34}$, which he obtained as follows: After trying 7 and 5, he finally decided that 34 would go into 163 four times. He wrote the 4 in his answer, multiplied 34 by 4 correctly, and, subtracting the result from 163, secured the correct remainder, 27. He divided 34, the last two digits in the dividend by 34, getting an answer of 1 with a remainder of 20. He then added the first remainder of 27 to the second

the requirements of the succeeding pupil. Moreover, among both pupils are to be found in the same classroom the conclusions so easily reached that the failing pupil merely stands in need of a repetition of the same kind of instruction under which the succeeding pupil has learned to thrive.

It is true that failure on the part of a pupil is evidence of trial and lack of success. Failure means much more, however; though in many respects it is a negative condition, it is also a very positive condition. The pupil who has failed to learn his arithmetic at any point not merely exhibits the lack of accomplishment, he also exhibits in a very positive way the accomplishment of something that is misleading and that prevents success. He has, to be sure, failed to learn what he should, but he has succeeded in learning something. Instead of putting forth the right kind of effort, he has put forth the wrong kind, instead of making real progress in his learning, he has made progress in the wrong direction. Stimulation to renewed effort or to an increase of effort may succeed only in driving him into more confusions than those already experienced. To return to our figure of the 'road of learning,' the failing pupil has moved off the main road and needs to be led to retract his steps to the point where he left it and there to take a new start.

For example, the pupil who has failed to learn division has not failed because he has had no instruction in division or no contacts with division. He has failed because he has had the wrong kind of instruction, or because he has been misled by the instruction he has had, or because he took the wrong view of division from the beginning. He has, to be sure, learned to give some correct answers to division questions when they are put to him, but he is never certain, because his memory fails him at times, or because the mere remembering of so many correct answers to division questions brings them into confusion. He has learned, let us say, that "two goes into ten five times," but since "goes into" has no meaning for him -- or for anyone else for that matter -- he must

The American Medical Association is a national organization of physicians and surgeons, and its primary purpose is to advance the science and art of medicine and surgery, and to protect the public health. It is a non-profit corporation, and its assets are held in trust for the benefit of the medical profession and the public. The Association is organized into various departments, including the Department of Education, the Department of Legislation, the Department of Public Health, and the Department of International Relations. It also maintains a number of journals and publications, and it holds annual meetings and conferences. The Association is a member of the World Medical Association, and it is in constant communication with the medical associations of other countries. Its efforts are directed towards the improvement of medical education, the advancement of medical research, and the promotion of the highest standards of medical practice. The Association is a powerful force in the medical world, and its influence is felt in every corner of the medical profession.

IV. THE AMERICAN MEDICAL ASSOCIATION'S POSITION ON THE ISSUE OF MEDICAL EDUCATION

The American Medical Association has long been a strong advocate of the traditional medical education system, which emphasizes the importance of a thorough grounding in the basic sciences and clinical experience. It has opposed any attempts to reform medical education, and it has been a major force in the defeat of various proposals for medical reform. The Association's position is based on the belief that the traditional system is the best way to ensure the highest quality of medical education, and that any changes would be detrimental to the public health.

A broad and deep understanding of the present and future of medical education is essential for any attempt to reform the system. The American Medical Association's position is based on a narrow and limited view of the problem, and it fails to take into account the many factors that are influencing the medical profession. The Association's opposition to medical reform is based on a number of misconceptions, and it is time to re-examine its position. The Association's position is based on the belief that the traditional system is the best way to ensure the highest quality of medical education, and that any changes would be detrimental to the public health. The Association's position is based on a narrow and limited view of the problem, and it fails to take into account the many factors that are influencing the medical profession. The Association's opposition to medical reform is based on a number of misconceptions, and it is time to re-examine its position.

The results of a recent survey, however, indicate that the medical profession is not as united as it once was. All the members of the Association are not in agreement on the issue of medical education, and the public's opinion is also changing. The Association's position is no longer as strong as it once was, and it is time to re-examine its position.

of thinking. The written advice which suggested practice at independent exercise and the system with which the practice were followed. However they have in the reader to supply an opportunity to the pupils concerned in the tests they conduct from recommendations which emphasized ideas that may be very valuable. The following extended quotations from Russell's discussion of the limitations of learning as a process of memorizing programs which it seems that the practice of a test may be very misleading. Concerning program literature regarding the nature of the process of learning and the process of learning arithmetic, he writes as follows:

A liberal generalization theory of that program in the development of ability in arithmetic may be adequately summarized by summarizing within the rule of the accuracy, or both the rule and the accuracy of performance. This theory as a material reality of that recognition of the learning process in arithmetic which views learning as the mere establishment of direct connections in the brain, between a kind of ideas, such as $7 + 5$, $12 + 3$, and $11 - 5$, and their corresponding answers, 12, 15, and 6. When learning in arithmetic is thought of in this way described by those who support the theory of facts, there are but two dimensions in the process, namely, rule and accuracy of performance. Methods of performance, being invariably the result of rote memory associations and therefore a constant factor, may be easily disregarded. Measurement of degree of development only in terms of rule and of accuracy is also in perfect accord with the conception of learning in arithmetic as drill, for drill is designed to set up the rote memory associations which make uniform all the processes and all the methods employed by children in dealing with numbers. Hence, according to this view, there is no variation in methods of dealing with numbers, degree of development in arithmetic are adequately measured when the two functions, rule and accuracy, are measured.

The individual analyses of the mental processes employed by pupils in dealing with visual concrete numbers, the additive combinations, and three-digit addition should go far toward convincing the reader that the views set forth in the

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The first race on September 22A. Everyman in the class of the test in addition, the teacher of Grade III A was asked to name the three pupils in two classes whom she regarded as the best in arithmetic. September 22A was designated as one of the three. He was the fourth pupil on the grade in completing the speed test in addition and the sixth pupil in completing the accuracy test. He made four errors and five errors respectively, in the two tests. In the practical tests he was asked to perform eight of the thirteen normal additions, he succeeded in five cases where the additions were within the limits of the simple additive combinations. The frequent case of counting is clear evidence of the fact that, while, from the stand-point of speed and accuracy in the group tests in addition, he appears to be above the average for the grade in the degree of his development in arithmetic ability, he is, as a matter of fact, very much retarded in his understanding of numbers and their relations. His speed and accuracy in addition are deceptive as an index of the true degree of his ability. He is simply rapid and accurate at a low stage of development. His teacher has incorrectly come to regard him as representative

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The following are additional examples, grouped in order to present various combinations of letters used in groups of four consonants and in consonant pairs in their combinations in their respective order. The examples, grouped under groups 1 - 2 and 3 - 4, 5 - 6, 7 - 8, 9 - 10, 11 - 12, etc., show many groups that combinations of 4 letters are possible from 2 - 4 with the vowel sound, where the combinations are given in the letter form. They may number to the 10 - 12 in order to reveal the systematic way to use vowels in the example 2.

It is very often independent to the conduct whether a pupil in the second grade begins at the age of 6 or the teachers, on being so far removed the correct manner that they themselves govern him. In such the degree in the preferred order rather than in the order as given from the age of 6 to 7. Consequently, in this example he made a great deal of work in the second grade in the preferred arrangement and thus secured further practice in the house last and the practice in the kindergarten. Later.



in the third grade, the pupil is given the example 8 and promptly announces the correct answer 16. The promptness and the accuracy of the computation convince the teacher that the pupil has met all possible requirements in the situation. Inquiry might reveal the fact that the pupil first combined $2 + 8$, because he preferred first to combine these digits, rather than $2 + 5$, $5 + 2$, $5 + 8$, or $8 + 5$ and, further, that he added the digits 2 and 8 in the order $8 + 2$, beginning at the bottom, skipping the 5 in the middle, and then coming back to the 5 to add it to the sum 10. The teacher's failure to determine the pupil's methods of procedure within the column results in the pupil's continuing to get practice (1) on

performed correctly the 2 and 4, or performed satisfactorily in these combinations, and 3 in the combinations of addition of digits in and those combinations 3 and 4, and those combinations for the pupil to recognize the patterns in combinations within the family combinations. These were the pupil's in the fourth grade. He is given the following examples (not having solutions):

- 8 (1)
- 6 (7)
- 6 (14)
- 2 (2)
- 8 (9)
- 7 (6)
- 4 (3)
- 9 (5)
- 5 (6)

The materials with the pupil were in great abundance in addition in the earlier grades are more thoroughly understood, and the children and the understanding of the pupil's work may be said to have reached the highest with his family combinations. These are especially true and learn that he has reached the stage in the course indicated by the materials in combinations of the right of the examples. The order of addition above was actually found in the course of a few days.

IV A To make certain that he had correctly reported his procedure, the same materials were given to him a second time, after an interval of a few minutes. The order of addition reported for the second trial differed from that reported for the first trial only by the fact that in the second trial he added the 6 after the 2 in a sequence of factors.

There is little doubt that this boy is one of the few of the teacher in the arithmetic class. His difficulties are not properly described, but his level of development indicated, when measurements are made only in terms of rate and accuracy in adding pairs of digits. His accuracy and his inaccuracy in dealing with long numbers are only indexes of a difficulty of a deeper and more fundamental sort. Furthermore, it is incorrect to speak of this boy as an arithmetic problem in the fourth grade, he was as truly a problem in the second grade and in the third grade, but his teachers, measuring his earlier degrees of development in terms of rate and accuracy, did not know it. His difficulty only came to the surface in the fourth grade, but it is of long growth and undoubtedly can be traced to his failure to develop efficient procedures and mechanics in earlier stages of arithmetic.*

* W. A. Brownell, *The Development of Children's Number Ideas in the Primary Grade* (The University of Chicago Press, Chicago, 1910).

the idea of percentage really means. He undoubtedly has suggested some of the difficulties that have been called to his attention. In effect, words he directs some of his energy away from computational procedure to the perception of thought. At first, these thought processes proceed somewhat slowly and uncertainly. Let the pupil now take a test in percentage, and his score will be lower than it was a few days or few weeks earlier. His performance seems unduly retrogressive instead of progressive, when in reality the pupil is forging ahead and making real progress for the first time. His lowered score, appearing as the opposite of progress, is really, in the particular instance mentioned, a necessary concomitant of progress.

Let us carry the illustration further. Suppose after the pupil has had instruction in percentage, he is given a test that includes exercises in both the first and the third cases of percentage. Let us suppose that he responds, as a great many pupils do, by multiplying in each exercise by the percent. His score will indicate a mastery of the first case and a lack of mastery of the third case; it will indicate perfection in the one case and a lack of understanding in the other. In reality the pupil's total performance should be interpreted as demonstrating the absence of any mastery of percentage. Unless the teacher goes back of the test score to the pupil's manner of performance, he will miss the real significance of the pupil's confusion of the third case of percentage with the first.

The writer has viewed the scores of a number of pupils in the fifth grade on a twenty-problem test involving the four fundamental processes. One of the papers showed that the pupil had added the numbers given in every one of the twenty problems. The score on the paper was 5. The score, taken at its face value, indicated that the pupil was able to deal correctly with exercises involving addition, but not with those involving subtraction, multiplication, and division. The pupil's method of performance, however, indicated that,

through the pupil and he was befuddled with regard to the real meaning of subtraction.

Again, pupils quickly learn to compute the areas of rectangles by following the simple rule of the rectangle. Their frequent inability to understand the meaning of area is not revealed by the errors they are able to produce on a test, even in those instances when they combine perimeters with area and find the dimension around a rectangular figure by multiplying the length by the width.

These errors, then, are merely indications of confusion or failure. A high score when accompanied by lack of understanding, and a low score when paralleled by progress. These errors are misleading and deceptive. We must go back to the test scores to the way they were produced if we wish to gain an idea of the standing of pupils.

VI. Methods of Diagnosis

With their diagnostic tools in the four fundamental processes, Howard and John propose a specific plan of diagnosis. Their plan places much emphasis upon a study of the pupil's methods of work and stress much a sharp distinction between diagnosing and testing that we quote at length:

Individual work. It is recommended that the diagnostic chart be used in the case of all pupils who are doing unsatisfactory work in arithmetic. The most economical method is to make a list of the names of the pupils whose work is to be analyzed and then to proceed systematically with the diagnoses, giving the other children in the group practice exercises or seat work until the diagnoses are completed. The diagnoses should be made individually and should cover only one of the four fundamental operations at a time. After assigning practice exercises or seat work to the class, the teacher should select a child whose work is to be diagnosed and sit down with him at her desk or at a table in the corner of the room. She should make the child feel as much at home as possible, since the success of the diagnosis depends on the extent to which the teacher becomes acquainted with the

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As the audit works, the immediate principal should be the design-
ing of the audit system which are concerned, as the system have
concerning the audit a particular as the system especially the ex-
amination. The audit system may be the first, at least for
the first few years as the system, as the audit system of the
project, the audit system. If the audit system have the other
concerning, it is authorized for the first as the audit procedure.
As a result of the examination, the immediate should have a clear
knowledge of the specific tasks which are responsible for the
project's first work.

Differences between diagramming and testing One particular distinction between the method of diagramming and the method of testing should be pointed out. After a test is given, the final score is computed, which indicates the grade of work which the pupil is doing. Ordinarily, attention centers simply on the score, which is used for purposes of classification. In the method of diagramming there is no final score. The procedure is used not for purposes of classification but rather for purposes of teaching. Consequently, the desired result is a clear

understanding on the part of the learner of what does the pupil does has work on rather than of some effective something very good. From this it is in the case the learner should not be satisfied simply with making the changes and not checking the change. The pupil should study carefully the characteristics of the pupil's work and therefore he has been trained the correct appropriate plan of remedial instruction.

It should be recognized here that not every student learned in the school may be regarded as a hard case. The pupil of average mental power is an illustration in point. In the preparation of the series, there is not an exceptional condition of pupils who are dealing with the fundamental operations, some of which they have experienced already for so well understood that an average pupil is needed. However, as a result of a study of some found that pupils were directed to some specific points, as much as case, the use of specific points which indicates that the pupil is following directions. Pupils of the kind are not just simply because they are less-competent and unreasoned habits of work should be changed in relation to the whole state of mind, which may be stated as the ability to use the fundamental operations accurately, rapidly, and with understanding.

VII Remedial Instruction

The third step in the commonly recommended technique of helping the retarded pupil, following the steps of testing and diagnosing, is that of providing remedial instruction at the points of failure. Testing is the means for judgment of discovering failure, and diagnosis is the means for judgment of discovering the causes of failure. Low scores in the other steps are treated as symptoms, and improper methods of work in the next are treated as the defects to be remedied. The third step is recommended with the idea that remedial instruction will be applied at the points where improper methods of work are discovered. The purpose of the remedial instruction is to substitute proper methods of work for improper methods.

* Burwell and John, *Op cit*, pp 123-124.

Remedial instruction is not the same as remedial in history. It is not the same as the remedial in the physical sciences and in mathematics. It is remedial in the sense of difficulty from accident, and as remedial for the physical sciences and mathematics it is remedial in the sense of difficulty from accident. It has the general virtue of calling the pupil's attention to specific and definite things to be overcome, and in general and definite things to be understood. However, as Remedial instruction

The use of such clinical methods in learning and diagnosis brings the clinician's method to the same point of view. It is found in other professions such as medicine. No one would think of diagnosing a disease by a physical examination when such methods involve the use of diagnosis as a mere symptom. In clinical medicine a few general rules are given. Clinical clinical methods are used to determine which, determine with accuracy as to the cause of the difficulty as well as the light of such information given the remedial instruction. The prevention of learning may be secured by higher levels of instruction and in the use of diagnostic techniques such as have been described and it is not necessary to try to correct the development of faulty habits and occasionally seek to determine the cause of difficulty apparent in pupils making little progress.

The peculiar virtue of remedial instruction applied at the point of difficulty is also its peculiar shortcoming. It tends to emphasize faults, and it is remedial rather than curative.

Remedial instruction as for the remedying of faults. The nature of the fault suggests the remedy. The remedy is devised and applied according to the fault. It is the fault that is to be overcome, so the fault tends to become the center of attention. To be sure, the correct procedure, which is to be substituted for the incorrect one, receives a large share of the emphasis, perhaps the major share; but the fault, as something to be abandoned or overcome, must needs

* I. J. Bruckner "Diagnosing pupil difficulties." *Journal of the National Education Association*, 21 April, 1932, p. 125.

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總編輯 吳曉波

We may summarize the characteristics of the foregoing organization in the elementary (1) that remedial instruction has only an immediate usefulness and a temporary value, (2) that remedial instruction is applied for minor defects, that does not strike deep at the roots of the defects, and (3) that the teaching pupil is generally in need of restorative instruction of the kind that will grow him more brought into the number system. In short, the foregoing pupil stands more in need of values of procedure than of methods of procedure.

It should be clear that the review of the work of preceding grades that usually features the first work of a school year does not provide the kind of narrative instruction needed by the failing pupil. Since this review is a repetition, the failing pupil may be hindered rather than helped by it, as the review may serve to implant still more deeply the wrong methods of work and the wrong attitude toward what is required in the subject. The failing pupil will not be helped by a repetition of the same kind of instruction and of the same kind of work that originally were responsible for his difficulties. He al-

The answer to the question is that the fraction $\frac{1}{2}$ is the same as the fraction $\frac{2}{4}$. The fraction $\frac{1}{2}$ is the same as the fraction $\frac{2}{4}$, because the numerator of the fraction is 1 and the denominator is 2, and the fraction $\frac{2}{4}$ has a numerator of 2 and a denominator of 4.

$$\frac{1}{2} = \frac{2}{4} = \frac{1}{2}$$

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Example: These fractions, arranged in order of size, the largest first, then the next largest, and so on:

$$\frac{1}{2}, \frac{2}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$$

The answer to the question is that the fraction $\frac{1}{2}$ is the same as the fraction $\frac{2}{4}$, because the numerator of the fraction is 1 and the denominator is 2, and the fraction $\frac{2}{4}$ has a numerator of 2 and a denominator of 4.

Example: These fractions, arranged in order of size, the largest first, then the next largest, and so on:

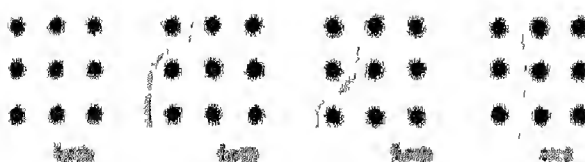
$$\frac{1}{2}, \frac{2}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$$

It is very easy to become confused about a fraction, but if you will remember about numerators and denominators, you need never be confused. Let us study why now as often confused and learn how to avoid being confused.



Let us divide a circle into 8 equal parts, and let us shade 3 of the parts. The shaded portion is $\frac{3}{8}$ of the circle.

Remember in thinking about what the fraction, $\frac{3}{8}$, tells us, we



When you had an arranged the objects and marked that 1000 objects are the same as 10 tens and 100, you wrote the answer as 10 tens and 0, or 10. The 10 is 10 tens number, and the place where 0 is written shows that the one of each is a unit of 10. The 10 is 10 tens number, and the place where 1 is written shows that the one of each of the 1 groups is 10.

Now we, when any quantity is written, the figure in each place shows number, and the place where it is written shows size. In the numbers, 1000, 100, 10, 1, the 1 shows number and the position shows that the one of the group is a thousand, the 1 shows number, and the position shows that the one of the group is a hundred, and so on.

The figure shows number, the position shows size.

A Lesson Showing the Importance of Position

All through your work in arithmetic you have learned how important it is to write numbers in their proper position. You have learned that, if a person is careful to write each number in its proper place, position will take care of everything else for him. Perhaps you remember that whenever you had something new to learn, like multiplying and dividing by tens, adding and subtracting decimals, and so on, you were told to write each number in its proper place and each part of the answer in its proper place. Why was there so much said about position, or place? Because position shows size, and it is necessary in our work in arithmetic not to mix sizes, but to keep the same sizes together.

Have you ever noticed how the grocer keeps the various sizes of his canned goods all sorted out and arranged on his shelves? For instance, he has all of his small-size cans of tomatoes together, all of his medium-size cans of tomatoes together, and all of his large-size cans together. Why do you suppose it is so important for the grocer to keep the various sizes of canned tomatoes sorted out so they will not be all mixed up on his shelves?

Let us add 522, 25, and 43

(a)	(b)
522	522
25	25
43	43
590	610

Why is (a) subtracted, and (b) corrected?

3
2

In (a) we have the paper all covered. In the end, 3, we have written 3 hundreds, 2 tens and 4 units and subtract. We get 5, but suddenly we see what error there is. We subtract 100. In subtracting, we must keep 5 hundreds of the same size together. We subtract 100 from 500. We have 400 left. We subtract 20 from 400. We have 380 left. We subtract 43 from 380. We have 337 left. Then we add a correction. We have exactly what was the result of the subtraction. We get 337.

Let us subtract 25 from 592

(a)	(b)
592	592
25	25
567	567

In (a) we have found the same. In (b) we have kept the numbers of the same size together.

A Lesson on the Size and Number of Divided Parts

In a common fraction it is easy to tell the numerator (number of parts) and the denominator (size of parts). In a decimal fraction it is just as easy to tell the numerator and denominator. In a common fraction we write both the numerator and the denominator. In a decimal fraction we write only the numerator, and we show the denominator by position.

In the decimal, 5, the 5 shows the number of parts, and its position in tenth's place shows the size of the parts.

In the decimal, 05, the 5 shows the number of parts, and its position in hundredth's place shows the size of the parts.

In the decimal, 672, the 6 shows the number of parts, and the position of the hundredth + parts shows the size of the parts.

In order to tell the size of the parts in a decimal fraction, we look at the position of the last figure on the right.

26 shows 2 tens, hundred + parts and 6 in hundredths + parts. 26 shows 2 tens and 6 hundredths. That means 2 tens in the same as 20 hundredths, 2 tens and 6 hundredths are 26 hundredths. We are told 26 is 26 hundredths. 26 is the number of parts, and the position of the 6 tells us the size.

26 shows 2 in tens + parts. We read 26 as 26 hundredths. 26 shows the number of parts, and the position of the 6 tells us the size.

626 shows 6 tens, 2 hundredths, and 6 thousandths. That means 6 tens in the same as 600 hundredths, and 2 hundredths in the same as 20 thousandths, 626 is read as 626 thousandths. 626 shows the number of parts, and the position of the 6 shows the size.

626 is read 626 thousandths. 626 shows the number of parts, and the position of the 6 shows the size.

Just as in the adding and subtracting of whole numbers, we must keep numbers of the same size together, so in the adding and subtracting of decimals, we must keep parts of the same size together.

Let us add 5, 42, and 626

(a)	(b)	
5	5	600
42	42	or 420
626	626	626
672	1 045	1 045

Why is (a) incorrect? Why is (b) correct?

Let us subtract .024 from .36.

(a)	(b)	
.36	.36	.360
.024	.024	or .024
.12	.336	.336

Why is (a) incorrect? Why is (b) correct? Remember Keep parts of the same size together. Why?

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CHAPTER XXV

OUTLINE OF THE COURSE IN ARITHMETIC

ANALYSIS

1. The grade-placement and sequence of the topics in arithmetic that preceding chapters have described are suggested in outline.

2. The development of number ideas by the pupil is briefly described.

3. A pupil's progress may be stated in terms of the general ideas he has developed. Ability to engage in computation is never an adequate measure of progress.

4. A pupil's progress may be stated in terms of his understanding of the number system.

5. The number system is universal in the possibilities of its use by one who understands it.

6. The arithmetic of the school has the single function of introducing the pupil to the meaning and use of the number system.

—————

The present chapter will undertake (1) to present a brief outline of the main topics in the course in arithmetic, indicating the grade in which each probably belongs, and (2) to bring together into a summary statement the theories, purposes, and points of view of our discussions to this point.

I. THE COURSE IN OUTLINE

The outline that follows shows grade-placement and sequence of topics. Of sequence one can be reasonably certain; of grade-placement there must, of course, be considerable doubt. The placement of topics follows closely the placement to be found in courses of study and textbooks in common use. If at any time it should appear that the placement

and a larger number for the dividend. The quotient is written directly for the dividend, but will have no for a multiplier and the product is given nothing from above.

After this is called separately for the investigation of the multiplication facts of multiplication and division until the fourth grade is reached and then the whole system of division is generally organized. Language, number, fractions, decimals and percentages. The fundamental and various language is most clearly indicated in the exercises. The reader is referred to the preceding chapters for suggestions about the relations of various language to number language.

Circle I and II

1. Circumference
2. The Study of Circumference to Ten
3. The Idea of Ten Circumference by Ten
4. Enlarging the Number Idea
 - a. Adding and Subtracting Tens
 - b. Higher-Place Circumference
 - c. Column Addition

D

5. Circumference by Ten (This is the circumference addition in D, and the corresponding subtraction ideas then corresponding multiplication ideas in Topic 4.)

No sharp line can be drawn between off the work of Circle I distinctly from the work of Circle II. It is suggested that Topics 1 to 3 as they relate to the groups to ten, be considered first-grade work and that Topics 4 and 5, including relations to and beyond ten and the preceding topics, be considered second-grade work. However, the work of the first grade includes no more than Topics 1 and 2 the work of the second grade must start, after the necessary review, with Topic 3.

Circle III

1. Review (an enlarged view of things already learned)
2. Tens in Addition and Multiplication (including carry-
ing)

2. Using Tools to Measure Length
3. Comparing and Ordering Lengths, Masses, and Capacities
4. Measuring Time and Temperature
5. Patterns

Grade 5

1. Review (an enlarged view of things already learned, to include a review of the uses of the operations of addition, subtraction, multiplication, and division)
2. Multiplying by Ten
3. Fractions and Decimals (relating the concepts, comparing and ordering, and adding)
4. Dividing by Ten
5. Fractions and Decimals (relating the concepts, comparing and ordering, and adding)
6. Fractions and Decimals (relating the concepts, comparing and ordering, and adding)
7. Comparing Fractions (relating the concepts and ordering)

Grade 6

1. Review (an enlarged view of things already learned, to include relating into a single topic for study the various uses of ten and position already learned)
2. Fractions (including the "three kinds of problems")
3. Decimals (including the "three kinds of problems")
4. Percentage (including the "three kinds of problems")
5. The Relations between Numbers
6. Measuring Surfaces (the rule of the rectangle)

Grade VI

1. Review (an enlarged view of things already learned: this includes a bringing together of the ideas and uses of ten, position, size, and number under the general topics of size and number of groups and of parts)

2. *Interplay of individual differences* for whom in daily activities of everyday experience are only formal exercises, leaving no lasting acquisition etc.
3. *Applying the social scientific method, abstracted, systematic etc.*
4. *Class and Exam*
5. *Measuring Intelligence and Learning*

II The Psychological Movement in Germany

The foregoing chapters deal with the development of the pupil's mental ideas in the elementary school grades. They point out how the developmental process in the systematic study of groups need from the developmental consideration through the orderly correspondence of each individual idea with another. They indicate the growth of ideas of relations between things that occur about through the process of building larger groups out of smaller ones and of dissolving larger groups into smaller ones, and through the use of meaningful terms to represent each individual idea in the relation to another. They describe and exemplify a large number of activities that at first glance seem to be very different and widely unrelated activities, but that are in reality intimately interrelated in that they all are useful in helping the pupil to develop and clarify his ideas of number.

III The Development of Logic

At the outset the pupil studies groups that he can see in detail, handle, and compare. He thus develops clear and usable ideas of such groups, and he thus learns to deal with them in a systematic way, and to think of each and to express each in its relation to the others. And he learns something else. He learns, in the dealing with the groups that are easy to handle, methods of dealing with groups that are not easy to handle. He can thus proceed in the later stages of his work to develop ideas of large groups through the extension

of the methods of procedure for learning as functions of small groups. Such methods are methods of presentation, of methods of arrangement, of methods of working, thinking, and expressing relations. There are just as significant as the ideas of number that develop with them through the study of groups.

The methods of procedure in which reference has been made are not merely methods. They are also ideas of procedure. In other words, the study of groups and of these methods of arrangement and society important methods through repetition. It also makes the methods understandable. The pupil not only learns to do what he is told, but also, because he first deals with groups that are easily comprehensible, understands and sees the significance of what he does.

IV The Review or 'Problems'

Special method is used to improve the methods and ideas of procedure, to clarify and enlarge them, and to make them familiar. Whether the ideas are those of addition, subtraction, multiplication, and division, or those of the fraction, percentage, the average, etc., the method is much the same. It is the method first, of presenting the idea, which already has been introduced, in a variety of familiar situations, each of which serves to make the idea stand out, and second, of giving the pupil practice in recognizing the idea. Such situations go by the name of 'problems,' and such activity is 'problem-solving,' so-called, at the earlier levels. The purpose is, as indicated, to make the ideas or methods familiar, so that they may be used in developing new number ideas and new relations and applications of number. At the later levels, after the ideas have become sufficiently familiar, the pupil can use them in his study of new personal, practical, business, and social situations, like buying and selling, budgeting, saving, investing, measuring, interest, insurance, and the like. These, likewise, are frequently presented in illus-

relative expressions which now go by the name of 'problema,' and demands an activity in dealing with them that is called 'problema-solving' at the later levels.

1. EXAMINING THE NUMBER SYSTEM

The whole the matter in another way, the pupil develops ideas of the numbers he has, learns to give special attention to two and finally, proceeds to deal with them as he dealt in the beginning with the numbers that gave rise to them. The pupil first learns to arrange the numbers he has in various ways, and then learns to arrange them 'just like usual,' until he arrives at the combinations of ten tens, or one hundred, and then proceeds to deal with hundreds, also, 'just like usual,' and so on up the scale. As the pupil proceeds, he learns to express his ideas of tens, hundreds, etc., by the use of the various symbols the world placed in different positions. The system of symbols which he has in expressing the relations of tens to tens, tens to hundreds, and so on, and his expression shows the fact that tens, hundreds, thousands etc. may be dealt with in all respects just as units are dealt with.

Supplementing the expressions almost relations between tens and hundreds and so on that are conveyed by the system of notation are the ideas of the part, or percent, which the pupil develops as ideas and expressions of relation. Such ideas the pupil learns to use as a means of gaining and of expressing a clear and usable idea of any given number as it relates to another number that may be relatively more familiar. Thus, the number that is relatively unfamiliar or about which a clearer idea is needed is expressed, in connection with one that may be known, in terms of a part, or a percent. The relatively unfamiliar number thus takes to itself the qualities of familiarity that attach to the known number. Thus, the idea of the part, or percent, becomes the device by which one balances one number against another, or expresses one number in terms of another.

§ 4. COMPUTATIONS

In the previous section we have seen how the pupil learns a large number of meaningless words and symbols are used in life's affairs, and many words and symbols are not used. It is not until the computations that are involved, both the words and the symbols, are used in the affairs of life, that a value for the pupil depends upon the situation in which they are learned. If learned in their relationship to the number system, both kinds of computations are of service in enlarging the pupil's ideas of numbers, large and small, and in establishing important relations between them. The computations of dealing by hundreds and thousands, for example, may never be useful to certain pupils in their later activities, but learning to deal by hundreds and thousands may serve to fix in mind the relations between the units as required and the units involved when one divides by tens and units. Similarly, the computations in the second and third cases of percentages may never be brought into use by certain pupils in dealing with the situations they will meet in later life, but the thinking that may be involved in connection with the computations mentioned may help to establish and to perfect the idea of percent as a meaningful and revealing expression of relations between numbers and may, in addition, lead to a better insight into the meanings of the numbers themselves. The computations mentioned may prove to be very useless, but the training they may give may be very useful. Computations may be learned as ends in themselves, or they may be learned as means to more important ends. When learned as ends, and without relation to the number system they may serve to illustrate, even the useful combinations may prove to be useless to the pupil, for he may not gain the insight sufficient to make them useful to him.

Let us turn to the physical education of children for an illustration of the relative importance of means and ends.

hand, the analysis may be very misleading if it is considered as a description of the arithmetic the pupil should learn. The description of our language suggests more the suggestion that a learning difficulty in the arithmetic the pupil is frequently called upon to learn comes from the fact that "computation" is generalized from "enumeration," that "application" follows as a naturalizing term used afterward, and that the number system is introduced and is left to fit as well as it can.

The teacher must, of course, make analysis, show the pupil cannot learn everything at once, but the teacher must be concerned with synthesis and with helping the pupil to be so concerned. As the pupil learns one thing, there another, he must be assisted in relating each to its proper place in the number system. It is the number system that is the end to be served.

Instead of there being four functions of arithmetic, there is only one, which, copying Bruner's term, we may call the "psychological." This is the function that has to do with orderly methods of arranging things or experiences into a common system that has come to be understandable. In other words, everything the pupil is required to learn in arithmetic may be thought of in terms of its relation to the number system. Computations must be learned, they may contribute to the development of the number system in the mind of the pupil, and what he has already learned of the system may make the computations clear. Information must be acquired, much information about numbers and number relations is acquired through computations, moreover, an understanding of number relations is very useful in gaining information about society and social situations. The transactions of business must be studied; a variety of business situations, not otherwise related, may be brought into relation as illustrations of a given way of thinking and expressing the relations between numbers. It is the number system that must develop in the mind of the pupil as a result of various methods of illustration, and it is the num-

under discussion. To say that the educational experience is no longer from the beginning is to overlook the fact that the experience of the transfer took place on a world not with that one. The world would not have been reached by a great world, and the world would be equipped with a great world. The world is a great world for the world, it is one of the great world of all the world.

The universality of number may be illustrated by the way that children make of counting when they first learn to count. They count everything that comes within the range of attention — the buttons on their shoes, the plates on the table, the chairs in the room, the pictures on the wall, the people on the street. They do this, however, only after they have learned to count. The object is not to make themselves countable, and they do not imagine their possibilities of orderly arrangement upon the child. He brings his counting ability, and he has gained it, as the child and applies it to them.

Similarly, the complex world does not imagine the number system upon the individual. The individual brings the system, if and when he has learned it, to the complex world and uses it as a means of bringing order out of complexity. When the system is learned, it attaches to anything and everything. No difficulty is experienced. The application of the number system, since it is learned, seems so easy and so natural that one is inclined to assume the system to be a part of one's natural endowments. Or the number system appears to be so much a matter of 'second nature' that the teacher is inclined, when he gives any thought to it, to assume that children will naturally grow into it. Perhaps this is the reason why reference to the number system is so conspicuously absent from the arithmetic of the schools.

Counting is easy, and its application is easy, once it is

¹ C. H. Judd. "Informational mathematics versus computational mathematics." P. 192. *The Mathematics Teacher*, 21: April, 1928, 187-196.

The first of these is the fact that the United States is a young nation, and that its history is a history of growth and development. The second is the fact that the United States is a nation of immigrants, and that its history is a history of the struggle for the rights of these immigrants. The third is the fact that the United States is a nation of free men, and that its history is a history of the struggle for the rights of these free men. The fourth is the fact that the United States is a nation of law, and that its history is a history of the struggle for the rights of these laws. The fifth is the fact that the United States is a nation of peace, and that its history is a history of the struggle for the rights of these peace. The sixth is the fact that the United States is a nation of justice, and that its history is a history of the struggle for the rights of these justice. The seventh is the fact that the United States is a nation of liberty, and that its history is a history of the struggle for the rights of these liberty. The eighth is the fact that the United States is a nation of equality, and that its history is a history of the struggle for the rights of these equality. The ninth is the fact that the United States is a nation of unity, and that its history is a history of the struggle for the rights of these unity. The tenth is the fact that the United States is a nation of progress, and that its history is a history of the struggle for the rights of these progress.

APPENDIX

A BIBLIOGRAPHY ON THE PSYCHOLOGY AND TEACHING OF ARITHMETIC

In the preparation of a bibliography two courses are open to the author. He may choose a list of 'selected references' to accompany each chapter, or he may group all his references at the end of his volume. The former course is especially appropriate when each chapter stands by itself more or less as a distinct entity, or when the materials of each are drawn more or less directly from definite portions of the available literature. The latter course is to be preferred when all the chapters unite in the development of a central theme.

The point of view developed in this book may be regarded as a composite of the author's personal reactions, unfavorable as well as favorable, to the available literature on arithmetic and its teaching. The point of view has also been shaped by his experience over a period of years, and it falls into an organized form that is somewhat at variance with the customary classification of the literature under such headings as 'history,' 'psychology,' 'methods,' 'drill,' 'problem-solving,' and the like. The exposition in each chapter is more an expression of personal opinion than an expression of a combination of views set forth in given references and is derived more directly from the central theme of the whole book than from the volumes, chapters, and articles of other writers. As a consequence, the writer has found it exceptionally difficult to name specific references that are directly responsible for his own views and difficult to assign a given reference to any given chapter.

In listing references, accordingly, it has seemed expedient to provide a separate section on 'Bibliography' that affords a medium for the listing both of those references

that there is considerable agreement for the majority of items found in a common method of procedure, and that the majority of items found in a common method of procedure are found in a common method of procedure.

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